Vektornormer:

1) $\|x\|_1 = \sum_{k=1}^{n} |x_k|$

2) $\|x\|_2 = \left[\sum_{k=1}^{n} |x_k|^2 \right]^{1/2}$

3) $\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|$

maxnormen

Matrisnormer:

1) $\|A\|_1 = \max_k \sum_{r=1}^{m} |a_{r,k}| \rightarrow$

största kolonnsumman

2) $\|A\|_2 = \max(\lambda(A^T A))^{1/2}$

3) $\|A\|_\infty = \max_r \sum_{k=1}^{n} |a_{r,k}|$

största radsumman

Konditionstal för $Ax=b$

(1) $Ax = b$

(2) $Ay = b + f$ störning i b

(1) − (2): $Ax - Ay = \cancel{b} - \cancel{b} - f$

$$A(x-y) = -f$$

Mult. till $\frac{A}{A}$:

$$x - y = -\frac{A \cdot A^{-1} f}{A}$$

$$\frac{x-y}{x} = -\frac{A \cdot A^{-1} f}{Ax}$$

Ax = b ($Ax=b$)

$$\boxed{\underbrace{\frac{\|x-y\|}{\|x\|}}_{\text{relativa felet}} \le \underbrace{\underbrace{\|A\| \cdot \|A^{-1}\|}_{K(A)} \cdot \frac{\|f\|}{\|b\|}}_{\text{uppskattning av relativa felet om } f \text{ är känd.}}}$$

Practical error bounds

$$Ax = b; \quad \delta x = \tilde{x} - x = \tilde{x} - A^{-1}b$$

Residual:

$$A\tilde{x} = b \to \text{resid.} \quad \boxed{r = A\tilde{x} - b} \Rightarrow$$

$$\Rightarrow A\tilde{x} = r + b$$

$$\tilde{x} = A^{-1}(r+b)$$

Then

$$\delta x = \tilde{x} - x = \tilde{x} - A^{-1}b =$$
$$= A^{-1}(r+b) - A^{-1}b =$$
$$= A^{-1}r + \cancel{A^{-1}b} - \cancel{A^{-1}b} =$$
$$= A^{-1}r$$

$$(*) \quad \boxed{\text{error} = \frac{\|\tilde{x} - x\|}{\|x\|} \le \|A^{-1}\| \frac{\|r\|}{\|x\|}}$$

This can be estimated using Hager's alg.

for $\|\cdot\|_\infty$

$$\|A^{-T}\|_1 = \|A^{-1}\|_\infty$$

Multiply and divide by $\frac{\|A\|}{\|A\|}$ inequality $(*)$ to get:

$$\boxed{\text{relative error} = \frac{\|\tilde{x} - x\|}{\|x\|} \le \underbrace{\|A^{-1}\| \cdot \|A\|}_{K(A)} \frac{\|r\|}{\|b\|}}$$

Perturbation theory

$$(1)\quad Ax = b \;;\quad \tilde{x} = \delta x + x$$

$$(2)\quad (A+\delta A)\,\tilde{x} = b + \delta b$$

$$(2)-(1) \Rightarrow \underbrace{A\tilde{x}}_{A(\delta x + x) = A\tilde{x}} + \delta A\,\tilde{x} - Ax = \cancel{b} + \delta b - \cancel{b}$$

$$A\delta x + \cancel{Ax} + \delta A\,\tilde{x} - \cancel{Ax} = \delta b$$

$$A(\delta x) + \delta A\,\tilde{x} = \delta b$$

want to estimate

$$A\,\delta x = \delta b - \delta A\,\tilde{x}$$

$$\delta x = A^{-1}(\delta b - \delta A\cdot\tilde{x})$$

$$\|\delta x\| \le \|A^{-1}\|\cdot\left(\|\delta b\| + \|\delta A\|\cdot\|\tilde{x}\|\right)$$

Divide and multiply to $\frac{\|A\|}{\|A\|}$ to get:

$$\underbrace{\frac{\|x-\tilde{x}\|}{\|x\|}}_{\text{relative error}} = \frac{\|\delta x\|}{\|x\|} \le \underbrace{\|A^{-1}\|\cdot\|A\|}_{K(A)\ \text{condition number of } A}\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)$$