MATHEMATICS Univ.of Gothenburg and Chalmers University of Technology
Examination in algebra: MMG500 and MVE 150, 2017-08-16.
No books, written notes or any other aids are allowed.
Telephone 031-772 5325.

1 Prove or disprove the following statements

a) Every group of order at most 4 is cyclic. 2p

b) Every group of order at most 6 is abelian 2p

2a) Define the characteristic of a ring. . 1p

2b). Determine the characteristic of the ring $2\mathbf{Z}$ of even integers. 1p

2c) Determine the characteristic of the ring $2\mathbf{Z} \times \mathbf{Z}_3$ 1p

2d). Determine the characteristic of the infinite direct product 1p

$\prod \mathbf{Z}_p = \mathbf{Z}_2 \times \mathbf{Z}_3 \times \mathbf{Z}_5 \times \ \ldots$ over all primes p.

3a) Explain why the non-zero elements in $\mathbf{Z}_{2017}$ form a multiplicative group. 1p

3b) Explain why the equation $x^n = 1$ cannot have more than n roots in $\mathbf{Z}_{2017}$. 2p

3c) Show by means of theorems in group theory that the equation $x^{32} = 1$ has exactly 32 roots in $\mathbf{Z}_{2017}$. 2p

(Note that $2016 = 2^5 3^3\, 7$ while 2017 is not divisible by any prime $p<45$.)

4 The sides of a cube are marked with one to six dots to form a die and two marked cubes give the same dice if and only if they are related by a rotational symmetry. Show that there are exactly 30 dice. 4p

(Only solutions based on group theory will receive points.)

5. Formulate and prove Lagrange's theorem. 4p

6. Show that any finite integral domain is a field. 4p

The theorems in Durbin's book may be used to solve exercises 1–4, but all claims that are made must be motivated.