Solutions to examination in algebra 2017-08-16.

MMG 500, MVE 150

1a. The assertion is false as $\mathbf{Z}_2 \times \mathbf{Z}_2$ is a group of order 4, which is not cyclic.

1b. The assertion is false as S_3 is a group of order 6, which is not abelian.

2a. The characteristic of a ring R is 0 if for some $r \in R$ all finite sums $r + \ldots + r \neq 0$.

Otherwise the characteristic of R is the smallest positive integer n such that $nr = 0$ for all $r \in R$.

2b. The characteristic of $R = 2\mathbf{Z}$ is 0 as all finite sums $2 + \ldots + 2 \neq 0$.

2c. The characteristic of $R = 2\mathbf{Z} \times \mathbf{Z}_3$ is 0 as all finite sums of copies of $(2, [0]_3) \in R$ are different from $(0, [0]_3)$

2d. Any multiple $n([1]_2, [1]_3, [1]_5, \ldots)$, $n \in \mathbf{N}$ of $([1]_2, [1]_3, [1]_5, \ldots) \in \prod \mathbf{Z}_p$ is $\neq 0$ as $n[1]_p \neq [0]_p$ for any prime p not dividing n. The ring is thus of characteristic 0.

3a. The non-zero elements in $\mathbf{Z}_{2017}$ form a multiplicative group as 2017 is a prime and $\mathbf{Z}_p$ is a field for any prime p. (Theorem in Durbin's book.)

3b. We have by a theorem in Durbin's book that a polynomial $f(x) \in K[x]$ over a field has at most n zeroes in K. Hence as $\mathbf{Z}_{2017}$ is a field $x^n - 1 = 0$ cannot have more than n zeroes in $\mathbf{Z}_{2017}$

3c. Let $U(\mathbf{Z}_{2017})$ be the multiplicative group of units in $\mathbf{Z}_{2017}$. As this group is of order $2016 = 2^5 3^3\, 7$, it contains a Sylow 2-subgroup G of order 32 in $U(\mathbf{Z}_{2017})$. By a corollary of Lagrange's theorem we have therefore that $x^{32} = 1$ for any $x \in G$. There are thus at least 32 elements $x \in \mathbf{Z}_{2017}$ with $x^{32} = 1$ and hence exactly 32 such elements by 3b.

4 The symmetry group of the cube consists of 24 rotations (see section 57 in Durbin's book). This group acts on the set of 6! =720 numberings by 1,2,3,4,5,6 dots of the sides of the cube. But none of these numberings will be fixed by any other rotation than the identity. We have therefore by Burnside's counting theorem (see op.cit.) that the number of non-equivalent dice is 720/24=30.

5. See Durbin's book.

6. See Durbin's book