MATHEMATICS
Univ. of Gothenburg and Chalmers University of Technology
Examination in algebra : MMG500 and MVE 150, 2018-08-22.
No books, written notes or any other aids are allowed.
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1) Let $F=\mathbf{Z}_2=\{0.1\}$ be the field of binary numbers and GL(2,F) be the multiplicative group of 2×2-matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with entries in F and determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$.

a) Determine the order of GL(2,F). 2p

b) Determine the normal subgroups of GL(2,F). 3p

2) Let S be the set of column vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with entries in $F=\mathbf{Z}_2$ and π: GL(2,F) $\times S \to S$ be the map which sends $\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$ to

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

a) Explain why π gives a group action of GL(2,F) on S. (You may use standard rules for matrix multiplication without proof.) 2p

b) Determine the orbit and stabiliser of $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S$ under this action, 2p

3) Let $\varphi: R_1 \to R_2$ be a homomorphism of rings and J an ideal in R_2. Show that $I=\varphi^{-1}(J)$ is an ideal of R_1. 4p

4) For primes p. let $\mathbf{Q}(\sqrt{p})$ be the set of all real numbers of the form $a+b\sqrt{p}$ for $a,b\in\mathbf{Q}$.

a) Show that $\mathbf{Q}(\sqrt{p})$ is a subfield of $\mathbf{R}$. 2p

b) Show that these fields are not isomorphic for different p. 2p

5. Let $* : G\times G \to G$ be an associative binary operation on a set G. 4p

a) Show that $(G, *)$ has at most one neutral element.

b) Show that each element of G has at most one inverse with respect to $*$.

6, Show that a polynomial of degree $n\geq 1$ over a field F 4p
has at most n roots in F.