Solutions to examination in algebra 2018-06-07.

MMG500 and MVE 150

1) Let $g \in G$ and $\phi(g) = ([a]_2, [b]_3, [c]_4, [d]_6)$. Then $\phi(g^{12}) = 12\phi(g) = (12[a]_2, 12[b]_3, 12[c]_4, 12[d]_6) = ([0]_2, [0]_3, [0]_4, [0]_6)$ such that $g^{12} \in \ker \phi$ for every $g \in G$.

2) We know by the fundamental theorem of abelian groups that there is a bijection between factorisations of 200 into prime powers (up to ordering) and isomorphism classes of abelian groups of order 200. There are six such factorizations of 200, namely 8×25, 4×2×25, 2×2×2×25, 8×5×5, 4×2×5×5 and 2×2×2×5×5.
There are thus six isomorphism classes of abelian groups of order 200. They are represented by $\mathbf{Z}_8\times\mathbf{Z}_{25}$, $\mathbf{Z}_4\times\mathbf{Z}_2\times\mathbf{Z}_{25}$, $\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_{25}$, $\mathbf{Z}_8\times\mathbf{Z}_5\times\mathbf{Z}_5$, $\mathbf{Z}_4\times\mathbf{Z}_2\times\mathbf{Z}_5\times\mathbf{Z}_5$ and $\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_5\times\mathbf{Z}_5$..

3) We use the subring criterion. First, $0 \in Z(R)$ as $0r=0=r0$ for all $r\in R$.
Next, let $z_1, z_2 \in Z(R)$ and $r\in R$. Then,
$(z_1+z_2)r = z_1r + z_1r = rz_1 + rz_2 = r(z_1+z_2)$
$(-z_1)r = -z_1r = -rz_1 = r(-z_1)$
$(z_1z_2)r = z_1(z_2r) = z_1(rz_2) = (rz_1)z_2 = (rz_1)z_2 = r(z_1z_2)$.
Hence z_1+z_2, $-z_1$, $z_1z_2 \in Z(R)$ for all $z_1, z_2 \in Z(R)$, which shows that $Z(R)$ is a subring of R

4) There are 24 rotations of the cube which form a group G acting on the set S of 2-colourings of the vertices. For $g\in G$, let $\Psi(g)$ be the number of 2-colourings preserved by g. The number of inequivalent 2-colourings of the vertices is then $\frac{1}{|G|}\sum_{g\in G}\Psi(g)$ by Burnside's lemma. Also, $\Psi(g)=2^{n(g)}$ for the number $n(g)$ of orbits of the action of $\langle g\rangle$ on the set V of vertices.

The 24 rotations in G are described in Example 57.3 in Durbin's book.

1. The identity.
2. Three 180° rotations around lines joining the centers of opposite faces.
3. Six 90° rotations around lines joining the centers of opposite faces.
4. Six 180° rotations around lines joining the midpoints of opposite edges.
5. Eight 120° rotations around lines joining opposite vertices.

For g of type 1, 2, 3, 4 resp.5 we have the following $\langle g\rangle$– orbits on V.

1. Eight orbits of length 1.
2. Four orbits of length 2.
3. Two orbits of length 4.
4. Four orbits of length 2.
5. Two orbits of length 1 and two orbits of length 3.

Hence $\Psi(g)=2^8$, 2^4 , 2^2 , 2^4 resp. 2^4. The number of inequivalent 2-colourings of the vertices is therefore

$$\tfrac{1}{24}(1\times2^8+3\times2^4+6\times2^2+6\times2^4+8\times2^4)=\tfrac{1}{24}(2^8+17\times2^4+6\times2^2)=\underline{23}$$

5) See sections 16 and 17 in Durbin's book.

6) See section 35 in Durbin's book.