(c) Explain why A is an extension of $\mathbb{R}$ (operation addition) by $\mathbb{R}^{\#}$ (operation multiplication).

(d) Give an example of a group that is an extension of $\mathbb{R}$ (operation addition) by $\mathbb{R}^{\#}$ (operation multiplication) and is not isomorphic to A.

23.20. (First Isomorphism Theorem) Assume that H and K are subgroups of a group G and $K \lhd G$.

(a) Prove that $HK = \{hk : h \in H \text{ and } k \in K\}$ is a subgroup of G, and that $K \lhd HK$. (Compare Problem 23.22.)

(b) Define $\theta : H \to HK/K$ by $\theta(h) = Kh$ for each $h \in H$. Verify that θ is a homomorphism of H onto HK/K.

(c) Verify that Ker $\theta = H \cap K$. Now explain why

$$\frac{H}{H \cap K} \approx \frac{HK}{K}.$$

23.21. (Second Isomorphism Theorem) Assume that H and K are subgroups of a group G, and that $K \lhd H$, $K \lhd G$, and $H \lhd G$. Prove that $H/K \lhd G/K$ and that $(G/K)/(H/K) \approx G/H$. [*Suggestion:* Consider $\theta : G/K \to G/H$ defined by $\theta(Kg) = Hg$. Verify that θ is well defined and is a homomorphism, and apply the Fundamental Homomorphism Theorem.]

23.22. Give an example to show that if A and B are subgroups of a group G, then $AB = \{ab : a \in A$ and $b \in B\}$ need not be a subgroup of G. [*Suggestion:* Try $G = S_3$. If $A \lhd G$ or $B \lhd G$, then AB is a subgroup; see Problem 23.20(a).]

OTES ON CHAPTER V

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