MATEMATIK	Datum: 2013-06-01	Tid: eftermiddag
GU, Chalmers	Hjälpmedel: inga	
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Lösningr till tenta i matematisk modellering, MMG510, MVE160
This exam is for students who learned the old version of the course in the year 2012 and earlier.

Answer first those questions that look simpler, then take more complicated ones etc. Good luck! **(4p)**

1. Formulate and give a proof to the theorem on instability of fixed points to autonomous systems of ODE by Liapunovs test functions. See the lecture notes on the homepage.

(4p)

2. Consider the following system of ODE:
$\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$, with a constant matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$. Give the evolution operator of the system. Sketch the phase protrait of the system.

Find all those initial vectors $\overrightarrow{r_0} = \overrightarrow{r}(0)$ that give bounded solutions to the system. **(4p)**

Matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$, has real eigenvalues $\lambda_1 = \sqrt{6} - 1, \lambda_2 = -\sqrt{6} - 1$, with different signs and eigenvectors: $v_1 = \begin{bmatrix} \frac{1}{2}\sqrt{6} + 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -\frac{1}{2}\sqrt{6} + 1 \\ 1 \end{bmatrix}$.

The origin is a stationary point called saddle point in this case.

General solution has the form $x(t) = C_1 v_1 \exp(\lambda_1 t) + C_2 v_2 \exp(\lambda_2 t)$.Solutions are bounded if and only if the initial vector $\overrightarrow{r_0}$ is parallel to v_2.

Trajectories tend to the line through the origin parallel to v_1 as $t \to +\infty$, and tend to the line through the origin parallel to v_2 as $t \to -\infty$.

The evolution operator is $\exp(tA)$ can be computed by using change of variables: $x = Uy$ with $U = (v_1, v_2) = \begin{bmatrix} \frac{1}{2}\sqrt{6} + 1 & -\frac{1}{2}\sqrt{6} + 1 \\ 1 & 1 \end{bmatrix}$. $U^{-1}AU = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = J$ and $A = UJU^{-1}$ implies

$$\exp(tA) = U \begin{bmatrix} \exp(t\lambda_1) & 0 \\ 0 & \exp(t\lambda_2) \end{bmatrix} U^{-1} \text{ with } U^{-1} = \begin{bmatrix} \frac{2\sqrt{6}+6}{6\sqrt{6}+12} & \frac{\sqrt{6}}{6\sqrt{6}+12} \\ -\frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} + \frac{1}{2} \end{bmatrix}.$$

Sylvesters method uses matrices $Q_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}$ and $Q_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$ such that $A = \lambda_1 Q_1 + \lambda_2 Q_2$, $Q_1 Q_2 = 0$, $Q_1^2 = Q_1$, $Q_2^2 = Q_2$ and the evolution operator is $\exp(At) = \exp(\lambda_1 t) Q_1 + \exp(\lambda_2 t) Q_2$.

In our case

$$Q_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = \frac{1}{(2\sqrt{6})} \left(\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} - (\sqrt{6} - 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} \sqrt{6}\left(-\frac{1}{12}\sqrt{6} + \frac{1}{6}\right) & \frac{1}{12}\sqrt{6} \\ \frac{1}{6}\sqrt{6} & \sqrt{6}\left(-\frac{1}{12}\sqrt{6} - \frac{1}{6}\right) \end{bmatrix};$$

$$Q_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1} = \frac{1}{(-2\sqrt{6})} \left(\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} - (-\sqrt{6} - 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} \sqrt{6}\left(-\frac{1}{12}\sqrt{6} - \frac{1}{6}\right) & -\frac{1}{12}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} & \sqrt{6}\left(-\frac{1}{12}\sqrt{6} + \frac{1}{6}\right) \end{bmatrix}$$

The stationary point is a saddle point. Trajectories ar hyperbolas that tend to straight lines through the origin parallel to the eigenvectors.

3. Consider the system of equations: $\begin{cases} x' = -x + y^2 \\ y' = -xy - x^2 \end{cases}$

Decide if the stationary point at the origin is asymptotically stable. **(4p)**

Check a test functuion $V(x,y) = x^2 + y^2$.

$\frac{1}{2}V'(t) = -x^2 + xy^2 - xy^2 - x^2y = -x^2(1-y)$

$V' \leq 0$ for $y < 1$ and is therefore a week Ljapunovs function. It implies that the origin is a stable fixed point. On the set $x = 0$ $V' = 0$.Checking the sign of the right hand sides of the equations we observe that $x' = y^2$ on the line $x = 0$ and therefore all trajectories cross this line. It implies that the origin is asymptotically stable fixed point.

4. Explain the notion Hopf bifurcation.

   Show that the system $\left\{ \begin{array}{l} x' = y - x^3 \\ y' = -x + \mu y - x^2 y \end{array} \right.$

   has a Hopf bifurcation for $\mu = 0$. **(4p)**

   The linearised system has matrix $\begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$, with complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\mu \pm \frac{1}{2}\sqrt{\mu^2 - 4}$.

   For $-2 < \mu < 0$ the system has a stable stationary point in the origin. For $0 < \mu < 2$ the system has an unstable stationary point in the origin. When $\mu = 0$ one cannot use linearization to make conclusions on stability. We try to use $V(x,y) = x^2 + y^2$ as Lyapunovs function for the system

   $$\left\{ \begin{array}{l} x' = y - x^3 \\ y' = -x - x^2 y \end{array} \right. \cdot \frac{d}{dt}(V) = \nabla V \cdot \begin{bmatrix} y - x^3 \\ -x - x^2 y \end{bmatrix} = -x^4 - x^2y^2 = -x^2\left(x^2 + y^2\right) \leq 0$$

   for $x = 0$ we observe that $x' = y \neq 0$ outside the origin. It implies that the trajectory goes out for the line $x = 0$

   everywhere exept the origin and the origin is an asymptotically stable stationary point.

   Show that the system $\left\{ \begin{array}{l} x' = \mu x + y \\ y' = -x - y^3 \end{array} \right.$

   has a Hopf bifurcation at $\mu = 0$. **(4p)**

5. Formulate Poincare-Bendixson theorem. Find a positively invariant set for the following system of ODE. Show that the system has at least one periodic solution.

   $$\left\{ \begin{array}{ll} x' = & -y/3 + x(1 - 3x^2 - y^2) \\ y' = & x + y(1 - 3x^2 - y^2) \end{array} \right. \qquad \textbf{(4p)}$$

   Poincare-Bendixson theorem. If C is a compact positively invariant set in the plane without fixed points, it must contain at least one periodic orbit.

   Multiply the first equation by $3x$ and the second equation by y and add:

   $\frac{1}{2}\left(3x^2 + y^2\right)' = \left(3x^2 + y^2\right)\left(1 - \left(3x^2 + y^2\right)\right)$

   The function $V(x,y) = 3x^2 + y^2$ satisfies the equation: $V'(t) = 2V(1 - V)$.

   We observe that $V(t)$ increases along trajectories of the system for $V < 1$ and V decreases for $V > 1$. It implies that the set $G = \left\{(x,y) : 0.5 \leq 3x^2 + y^2 \leq 2\right\}$ (an elliptic ring round the origin) is a positively invariant set.

   The same calculation shows that the origin is the only stationary point, because stationary point

   must satisfy the equation $V(1 - V) = 0$. By inserting $1 - 3x^2 - y^2 = 0$ into the equations one can see that points on the ellips $3x^2 - y^2 = 1$ are not stationary, because thay must at the same time be in the origin. It leaves the only fixed point in the origin.

   The corollary to Poincare Bendixson theorem states that in a compact positively invariant set without fixed points there must be at least one periodic solution.

Answer first those questions that look simpler, then take more complicated ones etc. Good luck!

Max 20 points

For GU: **VG:** 15 points; **G:** 10 points. For Chalmers: **5:** 17 points; **4:** 14 points; **3**: 10 points;

Total points for the course will be the average of points for the project (60%) and for this exam together with bonus points for home assingments (40%).