CHAPTER 1

Introduction

1.1. Dimensional Analysis

Exact solutions are rare in many branches of fluid mechanics, solid mechanics, motion, and physics because of nonlinearities, inhomogeneities, and general boundary conditions. Hence, engineers, physicists, and applied mathematicians are forced to determine approximate solutions of the problems they are facing. These approximations may be purely numerical, purely analytical, or a combination of numerical and analytical techniques. In this book, we concentrate on the purely analytical techniques, which, when combined with a numerical technique such as a finite-difference or a finite-element technique, yield very powerful and versatile techniques.

The key to solving modern problems is mathematical modeling. This process involves keeping certain elements, neglecting some, and approximating yet others. To accomplish this important step, one needs to decide the order of magnitude (i.e., smallness or largeness) of the different elements of the system by comparing them with each other as well as with the basic elements of the system. This process is called *nondimensionalization* or making the variables *dimensionless*. Consequently, one should always introduce dimensionless variables before attempting to make any approximations. For example, if an element has a length of one centimeter, would this element be large or small? One cannot answer this question without knowing the problem being considered. If the problem involves the motion of a satellite in an orbit around the earth, then one centimeter is very very small. On the other hand, if the problem involves intermolecular distances, then one centimeter is very very large. As a second example, is one gram small or large? Again one gram is very very small compared with the mass of a satellite but it is very very large compared with the mass of an electron. Therefore, expressing the equations in dimensionless form brings out the important dimensionless parameters that govern the behavior of the system. Even if one is not interested in approximations, it is recommended that one perform this important step before analyzing the system or presenting experimental data. Next, we give a few examples illustrating the process of nondimensionalization.

EXAMPLE 1

We consider the motion of a particle of mass m restrained by a linear spring having the constant k and a viscous damper having the coefficient μ, as shown in Figure 1-1. Using Newton's second law of motion, we have

$$m\frac{d^2u}{dt^2} + \mu\frac{du}{dt} + ku = 0 \tag{1.1}$$

where u is the displacement of the particle and t is time. Let us assume that the particle was released from rest from the position u_0 so that the initial conditions are

$$u(0) = u_0 \qquad \frac{du}{dt}(0) = 0 \tag{1.2}$$

In this case, u is the dependent variable and t is the independent variable. They need to be made dimensionless by using a characteristic distance and a characteristic time of the system. The displacement u can be made dimensionless by using the initial displacement u_0 as a characteristic distance, whereas the time t can be made dimensionless by using the inverse of the system's natural frequency $\omega_0 = \sqrt{k/m}$. Thus, we put

$$u^* = \frac{u}{u_0} \qquad t^* = \omega_0 t$$

where the asterisk denotes dimensionless quantities. Then,

$$\frac{du}{dt} = \frac{d(u_0u^*)}{dt^*}\cdot\frac{dt^*}{dt} = \omega_0 u_0 \frac{du^*}{dt^*}$$

$$\frac{d^2u}{dt^2} = \omega_0^2 u_0 \frac{d^2u^*}{dt^{*2}}$$

so that (1.1) becomes

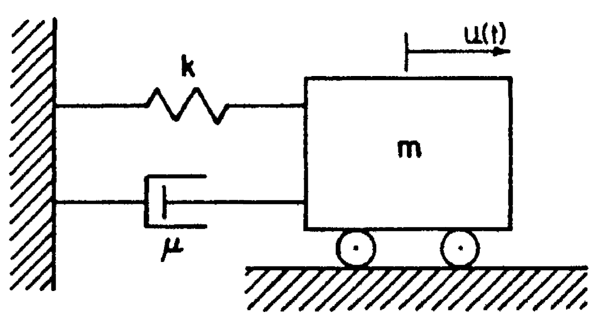

Figure 1-1. A mass restrained by a spring and a viscous damper.

$$m\omega_0^2 u_0 \frac{d^2u^*}{dt^{*2}} + \mu\omega_0 u_0 \frac{du^*}{dt^*} + ku_0 u^* = 0$$

Hence,

$$\frac{d^2u^*}{dt^{*2}} + \mu^* \frac{du^*}{dt^*} + \frac{k}{m\omega_0^2} u^* = 0$$

or

$$\frac{d^2u^*}{dt^{*2}} + \mu^* \frac{du^*}{dt^*} + u^* = 0 \tag{1.3}$$

where

$$\mu^* = \frac{\mu}{m\omega_0} \tag{1.4}$$

In terms of the above dimensionless quantities, (1.2) becomes

$$u^*(0) = 1 \qquad \text{and} \qquad \frac{du^*}{dt^*}(0) = 0 \tag{1.5}$$

Thus, the solution to the present problem depends only on the single parameter μ^*, which represents the ratio of the damping force to the inertia force or the restoring force of the spring. If this ratio is small, then one can use the dimensionless quantity μ^* as the small parameter in obtaining an approximate solution of the problem, and we speak of a lightly damped system. We should note that the system cannot be considered lightly damped just because μ is small; $\mu^* = \mu/m\omega_0 = \mu/\sqrt{km}$ must be small.

EXAMPLE 2

Let us assume that the spring force is a nonlinear function of u according to

$$f_{\text{spring}} = ku + k_2u^2 \tag{1.6}$$

where k and k_2 are constants. Then, (1.1) becomes

$$m\frac{d^2u}{dt^2} + \mu\frac{du}{dt} + ku + k_2u^2 = 0 \tag{1.7}$$

Again, using the same dimensionless quantities as in the preceding example, we have

$$mu_0\omega_0^2 \frac{d^2u^*}{dt^{*2}} + \mu u_0\omega_0 \frac{du^*}{dt^*} + ku_0u^* + k_2u_0^2u^{*2} = 0$$

or

$$\frac{d^2u^*}{dt^{*2}} + \mu^* \frac{du^*}{dt^*} + u^* + \epsilon u^{*2} = 0 \tag{1.8}$$

where

$$\mu^* = \frac{\mu}{m\omega_0} \quad \text{and} \quad \epsilon = \frac{k_2 u_0}{k} \tag{1.9}$$

The initial conditions transform as in (1.5). Thus, the present problem is a function of the two dimensionless parameters μ^* and ϵ. As before, μ^* represents the ratio of the damping force to the inertia force or the linear restoring force. The parameter ϵ represents the ratio of the nonlinear and linear restoring forces of the spring.

When we speak of a weakly nonlinear system, we mean that k_2u_0/k is small. Even if k_2 is small compared with k, the nonlinearity will not be small if u_0 is large compared with k/k_2. Thus, ϵ is the parameter that characterizes the nonlinearity.

EXAMPLE 3

As a third example, we consider the motion of a spaceship of mass m that is moving in the gravitational field of two fixed mass-centers whose masses m_1 and m_2 are much much bigger than m. With respect to the Cartesian coordinate system shown in Figure 1-2, the equations of motion are

$$m\frac{d^2x}{dt^2} = -\frac{mm_1Gx}{(x^2+y^2)^{3/2}} - \frac{mm_2G(x-L)}{[(x-L)^2+y^2]^{3/2}} \tag{1.10}$$

$$m\frac{d^2y}{dt^2} = -\frac{mm_1Gy}{(x^2+y^2)^{3/2}} - \frac{mm_2Gy}{[(x-L)^2+y^2]^{3/2}} \tag{1.11}$$

where t is the time, G is the gravitational constant, and L is the distance between m_1 and m_2.

In this case, the dependent variables are x and y and the independent variable is t. Clearly, a characteristic length of the problem is L, the distance between the two mass centers. A characteristic time of the problem is not as obvious. Since

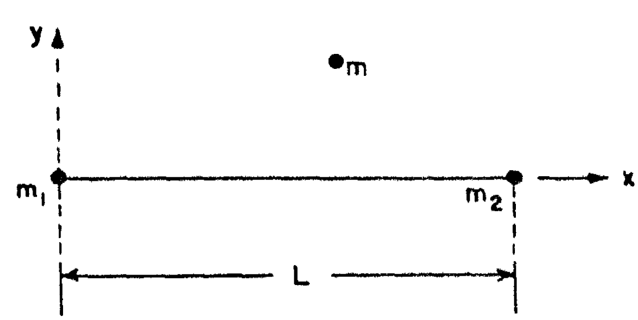

Figure 1-2. A satellite in the gravitational field of two fixed mass centers.

the motions of the masses m_1 and m_2 are assumed to be independent of that of the spaceship, m_1 and m_2 move about their center of mass in ellipses. The period of oscillation is

$$T = \frac{2\pi L^{3/2}}{\sqrt{G(m_1 + m_2)}}$$

so that the frequency of oscillation is

$$\omega_0 = L^{-3/2} \sqrt{G(m_1 + m_2)} \tag{1.12}$$

Thus, we use the inverse of ω_0 as a characteristic time. Then, we introduce dimensionless quantities defined by

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad t^* = \omega_0 t \tag{1.13}$$

so that

$$\frac{dx}{dt} = \frac{d(x^*L)}{dt^*}\frac{dt^*}{dt} = L\omega_0 \frac{dx^*}{dt^*} \qquad \frac{d^2x}{dt^2} = L\omega_0^2 \frac{d^2x^*}{dt^{*2}}$$

$$\frac{dy}{dt} = \frac{d(y^*L)}{dt^*}\frac{dt^*}{dt} = L\omega_0 \frac{dy^*}{dt^*} \qquad \frac{d^2y}{dt^2} = L\omega_0^2 \frac{d^2y^*}{dt^{*2}}$$

Hence, (1.10) and (1.11) become

$$mL\omega_0^2 \frac{d^2x^*}{dt^{*2}} = -\frac{mm_1GLx^*}{[L^2(x^{*2} + y^{*2})]^{3/2}} - \frac{mm_2GL(x^* - 1)}{[L^2(x^* - 1)^2 + L^2y^{*2}]^{3/2}}$$

$$mL\omega_0^2 \frac{d^2y^*}{dt^{*2}} = -\frac{mm_1GLy^*}{[L^2(x^{*2} + y^{*2})]^{3/2}} - \frac{mm_2GLy^*}{[L^2(x^* - 1)^2 + L^2y^{*2}]^{3/2}}$$

or

$$\frac{d^2x^*}{dt^{*2}} = -\frac{m_1G}{L^3\omega_0^2}\frac{x^*}{(x^{*2} + y^{*2})^{3/2}} - \frac{m_2G}{L^3\omega_0^2}\frac{(x^* - 1)}{[(x^* - 1)^2 + y^{*2}]^{3/2}} \tag{1.14}$$

$$\frac{d^2y^*}{dt^{*2}} = -\frac{m_1G}{L^3\omega_0^2}\frac{y^*}{(x^{*2} + y^{*2})^{3/2}} - \frac{m_2G}{L^3\omega_0^2}\frac{y^*}{[(x^* - 1)^2 + y^{*2}]^{3/2}} \tag{1.15}$$

Using (1.12), we have

$$\frac{m_1G}{L^3\omega_0^2} = \frac{m_1}{m_1 + m_2} \qquad \frac{m_2G}{L^3\omega_0^2} = \frac{m_2}{m_1 + m_2}$$

Hence, if we put

$$\frac{m_2}{m_1 + m_2} = \epsilon \qquad \text{then} \qquad \frac{m_1}{m_1 + m_2} = 1 - \epsilon \tag{1.16}$$

and (1.14) and (1.15) become

$$\frac{d^2x^*}{dt^{*2}} = -\frac{(1-\epsilon)x^*}{(x^{*2}+y^{*2})^{3/2}} - \frac{\epsilon(x^*-1)}{[(x^*-1)^2+y^{*2}]^{3/2}} \tag{1.17}$$

$$\frac{d^2y^*}{dt^{*2}} = -\frac{(1-\epsilon)y^*}{(x^{*2}+y^{*2})^{3/2}} - \frac{\epsilon y^*}{[(x^*-1)^2+y^{*2}]^{3/2}} \tag{1.18}$$

Therefore, the problem depends only on the parameter ϵ, which is usually called the reduced mass. If m_1 represents the mass of the earth and m_2 the mass of the moon, then

$$\epsilon \approx \frac{\frac{1}{80}}{1+\frac{1}{80}} = \frac{1}{81}$$

which is small and can be used as a perturbation parameter in determining an approximate solution to the motion of a spacecraft in the gravitational field of the earth and the moon.

EXAMPLE 4

As a fourth example, we consider the vibration of a clamped circular plate of radius a under the influence of a uniform radial load. If w is the transverse displacement of the plate, then the linear vibrations of the plate are governed by

$$D\nabla^4 w - P\nabla^2 w - \rho\frac{\partial^2 w}{\partial t^2} = 0 \tag{1.19}$$

where t is the time, D is the plate rigidity, P is the uniform radial load, and ρ is the plate density per unit area. The boundary conditions are

$$\begin{aligned} w &= 0 \quad \frac{\partial w}{\partial r} = 0 \quad \text{at } r = a \\ w &< \infty \qquad\qquad\quad \text{at } r = 0 \end{aligned} \tag{1.20}$$

In this case, w is the dependent variable and t and r are the independent variables. Clearly, a is a characteristic length of the problem. The characteristic time is assumed to be T and it is specified below. Then, we define dimensionless variables according to

$$w^* = \frac{w}{a} \quad r^* = \frac{r}{a} \quad t^* = \frac{t}{T}$$

Hence,

$$\frac{\partial w}{\partial r} = \frac{\partial(aw^*)}{\partial r^*}\frac{dr^*}{dr} = \frac{\partial w^*}{\partial r^*}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial(aw^*)}{\partial \theta} = a\frac{\partial w^*}{\partial \theta}$$

$$\frac{\partial w}{\partial t} = \frac{\partial(aw^*)}{\partial t^*}\frac{dt^*}{dt} = \frac{a}{T}\frac{\partial w^*}{\partial t^*}$$

Since

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(1.19) becomes

$$\frac{D}{a^3}\left(\frac{\partial^2}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial}{\partial r^*} + \frac{1}{r^{*2}}\frac{\partial^2}{\partial \theta^2}\right)^2 w^* - \frac{P}{a}\left(\frac{\partial^2}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial}{\partial r^*} + \frac{1}{r^{*2}}\frac{\partial^2}{\partial \theta^2}\right) w^* - \frac{\rho a}{T^2}\frac{\partial^2 w^*}{\partial t^{*2}} = 0$$

or

$$\frac{D}{a^2 P}\nabla^{*4} w^* - \nabla^{*2} w^* - \frac{\rho a^2}{PT^2}\frac{\partial^2 w^*}{\partial t^{*2}} = 0 \qquad (1.21)$$

We can choose T to make the coefficient of $\partial^2 w^*/\partial t^{*2}$ equal to 1, that is, $T = a\sqrt{\rho/P}$. Then, (1.21) becomes

$$\epsilon\nabla^{*4} w^* - \nabla^{*2} w^* - \frac{\partial^2 w^*}{\partial t^{*2}} = 0 \qquad (1.22)$$

where

$$\epsilon = \frac{D}{a^2 P} \qquad (1.23)$$

In terms of dimensionless quantities, the boundary conditions (1.20) become

$$\begin{aligned} w^* &= \frac{\partial w^*}{\partial r^*} = 0 && \text{at} \quad r^* = 1 \\ w^* &< \infty && \text{at} \quad r^* = 0 \end{aligned} \qquad (1.24)$$

Therefore, the problem depends on the single dimensionless parameter ϵ. If the radial load is large compared with D/a^2, then ϵ is small and can be used as a perturbation parameter.

EXAMPLE 5

As a final example, we consider steady incompressible flow past a flat plate. The problem is governed by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.25}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1.26}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{1.27}$$

$$\begin{aligned} u = v = 0 \qquad & \text{at} \quad y = 0 \\ u \to U_\infty, v \to 0 \qquad & \text{as} \quad x \to -\infty \end{aligned} \tag{1.28}$$

where u and v are the velocity components in the x and y directions, respectively, p is the pressure, ρ is the density, and μ is the coefficient of viscosity.

In this case, u, v, and p are the dependent variables and x and y are the independent variables. To make the equations dimensionless, we use L as a characteristic length, where L is the distance from the leading edge to a specified point on the plate as shown in Figure 1-3, and use U_∞ as a characteristic velocity. We take ρU_∞^2 as a characteristic pressure. Thus, we define dimensionless quantities according to

$$u^* = \frac{u}{U_\infty} \qquad v^* = \frac{v}{U_\infty} \qquad p^* = \frac{p}{\rho U_\infty^2} \qquad x^* = \frac{x}{L} \qquad y^* = \frac{y}{L}$$

Then,

$$\frac{\partial u}{\partial x} = \frac{\partial(U_\infty u^*)}{\partial x^*}\frac{dx^*}{dx} = \frac{U_\infty}{L}\frac{\partial u^*}{\partial x^*} \qquad \frac{\partial u}{\partial y} = \frac{U_\infty}{L}\frac{\partial u^*}{\partial y^*} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{U_\infty}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

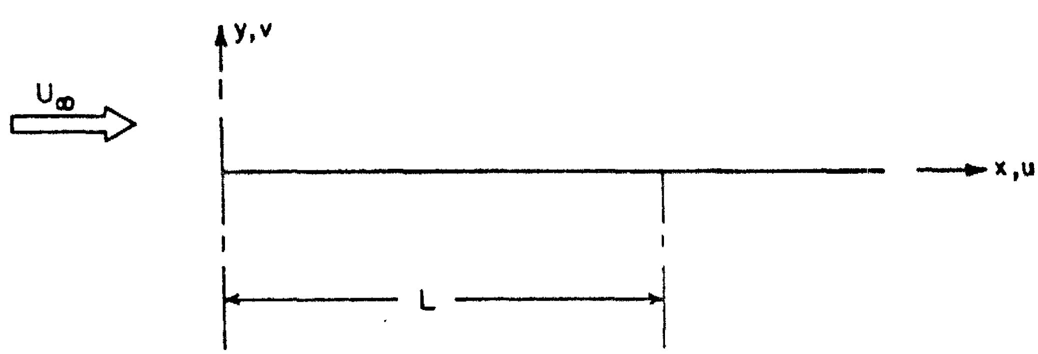

Figure 1-3. Flow past a flat plate.

$$\frac{\partial v}{\partial x} = \frac{U_\infty}{L}\frac{\partial v^*}{\partial x^*} \qquad \frac{\partial v}{\partial y} = \frac{U_\infty}{L}\frac{\partial v^*}{\partial y^*} \qquad \frac{\partial^2 v}{\partial x^2} = \frac{U_\infty}{L^2}\frac{\partial^2 v^*}{\partial x^{*2}} \qquad \frac{\partial^2 v}{\partial y^2} = \frac{U_\infty}{L^2}\frac{\partial^2 v^*}{\partial y^{*2}}$$

$$\frac{\partial p}{\partial x} = \frac{\rho U_\infty^2}{L}\frac{\partial p^*}{\partial x^*} \qquad \frac{\partial p}{\partial y} = \frac{\rho U_\infty^2}{L}\frac{\partial p^*}{\partial y^*}$$

Hence, (1.25) through (1.28) become

$$\frac{U_\infty}{L}\frac{\partial u^*}{\partial x^*} + \frac{U_\infty}{L}\frac{\partial v^*}{\partial y^*} = 0 \tag{1.29}$$

$$\frac{\rho U_\infty^2}{L} u^* \frac{\partial u^*}{\partial x^*} + \frac{\rho U_\infty^2}{L} v^* \frac{\partial u^*}{\partial y^*} = -\frac{\rho U_\infty^2}{L}\frac{\partial p^*}{\partial x^*} + \frac{\mu U_\infty}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{1.30}$$

$$\frac{\rho U_\infty^2}{L} u^* \frac{\partial v^*}{\partial x^*} + \frac{\rho U_\infty^2}{L} v^* \frac{\partial v^*}{\partial y^*} = -\frac{\rho U_\infty^2}{L}\frac{\partial p^*}{\partial y^*} + \frac{\mu U_\infty}{L^2}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{1.31}$$

$$\begin{aligned} &u^* = v^* = 0 && \text{at } y^* = 0 \\ &U_\infty u^* \to U_\infty,\ v^* \to 0 && \text{as } x^* \to -\infty \end{aligned} \tag{1.32}$$

Equations (1.29) through (1.32) can be rewritten as

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{1.33}$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{R}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{1.34}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{R}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{1.35}$$

$$u^* = v^* = 0 \qquad \text{at} \qquad y^* = 0 \tag{1.36}$$

$$u^* \to 1 \qquad v^* \to 0 \qquad \text{as} \qquad x^* \to -\infty \tag{1.37}$$

where

$$R = \frac{\rho U_\infty L}{\mu} \tag{1.38}$$

is called the Reynolds number.

Equations (1.33) through (1.37) show that the problem depends only on the dimensionless parameter R. For the case of small viscosity, namely μ small compared with $\rho U_\infty L$, R is large and its inverse can be used as a perturbation parameter to determine an approximate solution of the present problem. This process leads to the widely used boundary-layer equations of fluid mechanics. When the flow is slow, namely $\rho U_\infty L$ is small compared with μ, R is small and it can be