Home assignment I - in ODE and Mathematical modeling.
Linear theory for pendulum with oscillating pivot.

You are encouraged to work in small groups of 2-3 people, but each of you must write an own report (handwritten or in TEX) including: ***analytical work, theoretical argumentation, results and their interpretation after the plan below,*** *and names of the group members.* ***Send Matlab codes with clear comments by e-mail.*** *Time for the work is 2,5 weeks: good to be ready on Monday the 6-th of May, because one more assignment follows. Grades for your reports will contribute 30% to the final marks. You are welcome to contact the teacher in his office or by e-mail to pose questions.*

1. Consider the equation for pendulum with vertically oscillating pivot:

$$l\ddot{\theta} = -\left[g + \ddot{a}_y(t)\right]\sin\theta,$$

with harmonic exitation $a_y(t) = A\cos(\omega t)$. Vertical oscillations can, depending on the amplitude A and the frequency ω, make the upper position stable or the downward position unstable. Read some literature about pendulum (Wikipedia; Landau, Livshitz "Mechanics", etc).

2. Rewrite the equation in the non-dimensional form: in particular change the time scale t so that the exitation frequency will be unit: $\omega t \longmapsto t$.

3. Transform the non-dimensional equation for pendulum to a system of two first order equations.

Write a program in Matlab that solves this system of equations and illustrates solutions by parametrized integral curves $(\theta(t), \dot{\theta}(t))$ in the phase plane (as on pages 68-70 in the book):

4. Compute several solutions starting close to the fixed points $\theta = 0$, $\dot{\theta} = 0$ and $\theta = \pi$, $\dot{\theta} = 0$, corresponding to the stationary down and up positions of the pendulum.Illustrate these solutions by curves in the phase plane and check if they stay close to the fixed points.

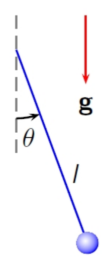
5. Linearize the original equation around the fixed points $\theta = 0$ and $\theta = \pi$. You will get two similar Mathieu equations that describe stability of these fixed points with different signs in the term with gravitational force. The sign in front of the exitation term actually does not matter:

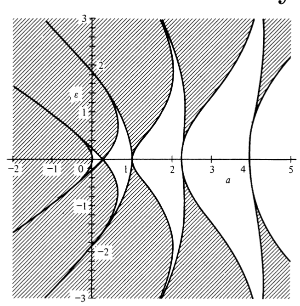
$$\ddot{y} + (\pm a \mp 2\varepsilon\cos t)\, y = 0.$$

Investigate stability of the corresponding linear systems of equations for various values of the amplitude ε and the frequency parameter a. Use Floquet theory for the Hill equation. It will include the following steps:

6. Learn Theorem 3.15 and Theorem 3.19 in §3.6 in the book.

Write a Matlab program to create a stability diagram as Fig.3.6 in the book or the





picture here: Stability diagram for the Mathieu equation.

Choose an array (rather large) of points (a, ε) in the parameter plane and make a loop in your program so that for each pair of parameters you do the following:

7. Compute numerically the monodromy matrix for the system of equations.

8. Calculate eigenvalues of the monodromy matrix and make conclusions about the stability of the system of equations for the particular pair of parameters. Put a marker into corresponding point (a, ε) in the plain of parameters. This calculations is clever to organize as a loop through chosen array of points so that at the end of the loop you get a stability diagram.

9. Check which part of the stability diagram for the linear Mathieu equation agrees with the stability of particular solutions of the original non-linear model for the pendulum.

In addition you can also try to carry out a more refined analysis of stability:

10. Find critical values of parameters (curves) in the parameter plane corresponding to transition from stability to instability or opposite. You can use here the Matlab *fzero* solver for non-linear equations.

11. What kinds of solutions correspond to these critical values of parameters?

12. At which particular points stability zones meet at the axis $\varepsilon = 0$?