ODE and modelling. Chapter I. 1.

Classification of differential equations.

$$F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$$

x - unknown function in $C^{(k)}(J)$, $J \subset \mathbb{R}$

$F \in C(U)$; $U \subset \mathbb{R}^{k+2}$ - open subset.

$$x^{(k)} = \frac{d^k x(t)}{dt}, \quad k \in \mathbb{N}_0$$

Notation $C^k(U, V)$ - set of functions with k continuous derivatives from U to V.

Standart form of equation:

$$x^{(k)} = f(t, x, x^{(1)}, \dots x^{k-1})$$

(can be transformed to this form at least locally if $\frac{\partial F}{\partial y_k}(t, y) \neq 0$

Classification: order, autonomous, non-autonomous, linear, non-linear, homogeneous, non-homogeneous, first order systems. Including time variable to dependent variables transforms non-autonomous system to autonomous.

§1.3 First order autonomous scalar eq.

Initial value problem: IVP.

(1.20) $\dot{x} = f(x)$, $x(0) = x_0$; $f \in C(\mathbb{R})$

$\Phi(t, x_0)$ - solution with initial data x_0.

Solution $\Psi(t)$ with $\Psi(t_0) = x_0$ is given by a shift: $\Psi(t) = \Phi(t - t_0, x_0)$.

If $f(x_0) \neq 0$ =>

$$\int_{x_0}^{x} \frac{1}{f(y)} dy = \int_0^t \frac{\dot{x}(s)\, ds}{f(x(s))} = \int_0^t ds = t$$

Let $F(x) = \int_{x_0}^{x} \frac{1}{f(y)} dy$

$F(x)$ is strictly monotone near x_0, because $f(x_0) \neq 0$ and continuous. $\Rightarrow$ F has inversion in a neighbourhood of x_0 that is solution to IVP. $\phi(t) = F^{-1}(t)$; $\phi(0) = F^{-1}(0) = x_0$

<u>Maximal interval of existence.</u>

If $f(x_0) > 0 \Rightarrow f > 0$ on an interval $[x_1, x_2]$ around x_0, by continuity.

Let $T_+ = \lim_{x \to x_2} F(x) \in (0, +\infty]$

(exists because $F(x)$ is monotone)

$T_- = \lim_{x \to x_1} F(x) \in [-\infty, 0)$

Implies: $\phi \in C^1(T_-, T_+)$ and $\lim_{t \to T_+} \phi(t) = x_2$

$\lim_{t \downarrow T_-} \phi(t) = x_1$.

$\Rightarrow$ ϕ exists for all $t \geq 0$ $(t < 0)$ $\Leftrightarrow$

$T_+ = \int_{x_0}^{x_2} \frac{dy}{f(y)} = +\infty$ (f - not integrable near x_2)

$\phi(t) \exists \ \forall t < 0 \Leftrightarrow \frac{1}{f(x)}$ is not integrable near x_1

$T_- = \int_{x_1}^{x_0} \frac{1}{f(y)} dy = -\infty$

Example. $\dot{x} = x$, $\int_{x_0}^{x} \frac{1}{x}\,dx = \int dt$ 3

$\phi(t) = x_0 e^t$ $\quad \ln|x| - \ln|x_0| = t$

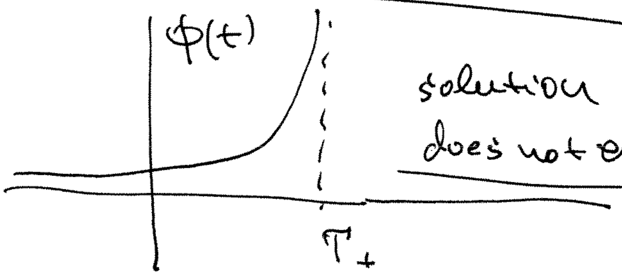
$x / x_0 = e^t$

Example $f(x) = x^2$; $F(x) = \int_{x_0}^{x} \frac{1}{x^2}\,dx =$

$$F(x) = \frac{1}{x_0} - \frac{1}{x} \quad \Rightarrow \quad T_+ = \frac{1}{x_0}$$

$$\boxed{\phi(t) = \frac{x_0}{1 - x_0 t}} \qquad \cancel{t} \rightsquigarrow \frac{1}{x_0} - t = \frac{1}{x}$$

$$\frac{1}{x} = \frac{1 - x_0 t}{x_0} \Rightarrow \boxed{\phi(t) = \frac{x_0}{1 - x_0 t}}$$



solution does not exist for $t \geq T_+$: blows up infinite time.

Fixed points: $f(x_0) = 0$

$\Rightarrow \phi(t) \equiv x_0$ is a solution

Is it unique?

If $\left| \int_{x_0}^{x_0 + \varepsilon} \frac{dy}{f(y)} \right| < \infty \Rightarrow \exists$

$$\varphi(t) = F^{-1}(t) \; ; \; F(x) = \int_{x_0}^{x} \frac{dy}{f(y)}$$

that is different from $\phi(t)$!

Example of non-uniqueness follows.

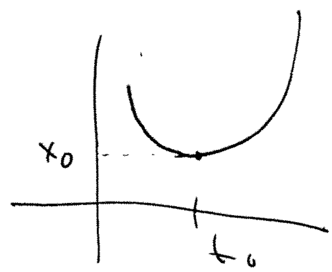
$\dot{x} = \sqrt{|x|}$; - symmetrical with respect to $x = 0$.

Consider $x \geq 0$.

$$\int_{x_0}^{x} \frac{dx}{\sqrt{x}} = \int_{t_0}^{t} dt \; ; \qquad 2\sqrt{x} - 2\sqrt{x_0} = t - t_0$$

$$\sqrt{x} = \sqrt{x_0} + \frac{t-t_0}{2} \; ;$$

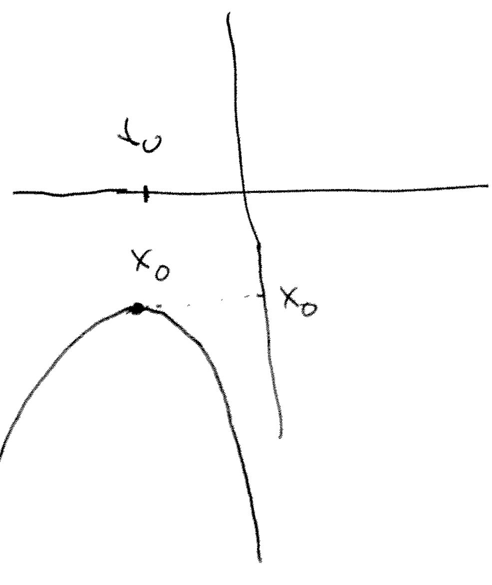
$$x = \left(\sqrt{x_0} + \frac{t-t_0}{2}\right)^2$$



$x \leq 0$ - case

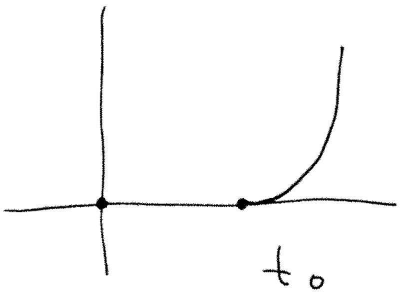
$$\int_{x_0}^{x} \frac{dx}{\sqrt{-x}} = \int_{t_0}^{t} dt \;\Rightarrow\; \sqrt{-x} = \sqrt{-x_0} + \frac{t-t_0}{2}$$

$$x = -\left(\sqrt{-x_0} + \frac{t-t_0}{2}\right)^2$$



4a)

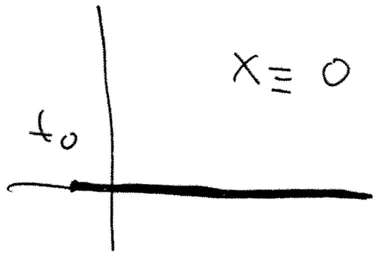
$X_0 = 0$

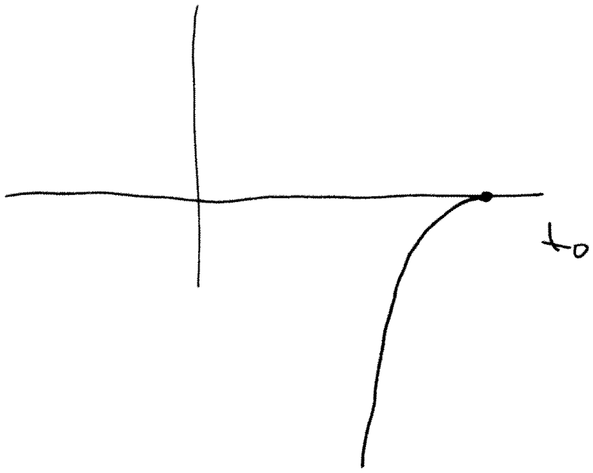


$$x = \left(\frac{t-t_0}{2}\right)^2$$

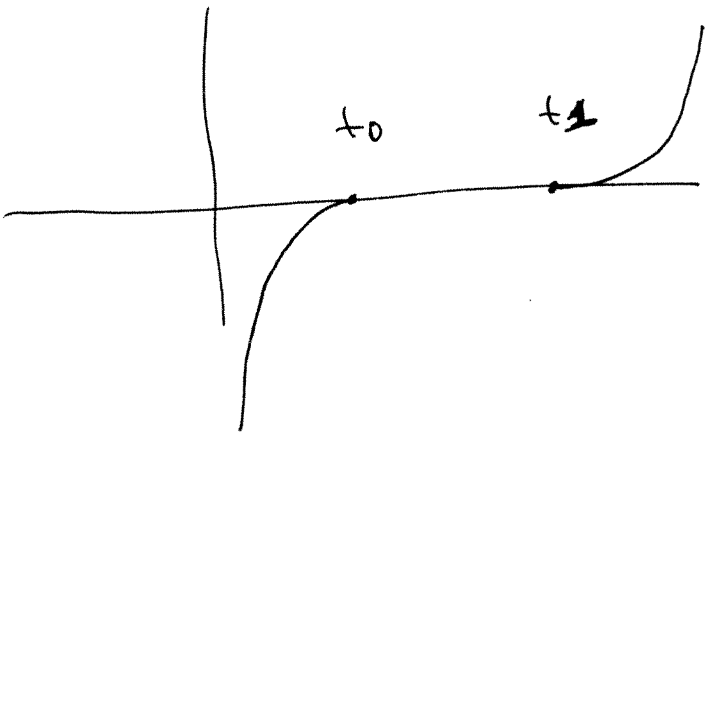
$x := 0$

$X \equiv 0$





$$x = -\left(\frac{t-t_0}{2}\right)^2$$



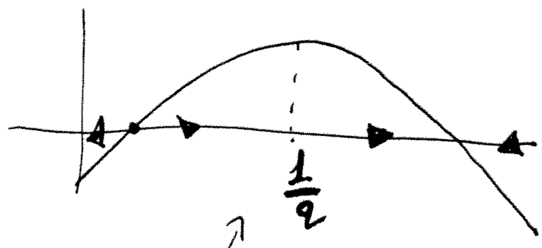
$$\tilde{\varphi}(t) = \begin{cases} -\left(\frac{t-t_0}{2}\right)^2, & t < t_0 \\ 0 & t_0 \le t \le t_1 \\ \frac{(t-t_1)^2}{4} & t_1 \le t \end{cases}$$

Qualitative analysis for 1-d systems

Example. Logistic growth model with harvest h.

$\dot{x}(t) = (1-x)x - h$

Look first at $f(x) = (1-x)x - h$



directions of movement

for $0 < h < \frac{1}{4}$ => there are two roots

$x_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1-4h}\right)$. At $h = \frac{1}{4}$ a bifurcation occurs

Two roots coincide

$h > \frac{1}{4}$ => no zeros and $\varphi(t) \xrightarrow[t\to+\infty]{} -\infty$.

Complete qualitative picture without solving the equation.

Lemma 1.1. $x' = f(x)$ 1-d., $f \in C(\mathbb{R})$ (solutions unique. suppose it here)

(i) If $f(x_0) = 0$; $x(t) = x_0 \ \forall t$.

(ii) if $f(x_0) \neq 0$ => $x(t)$ converges to the first zero left ($f(x_0) < 0$) respectively right ($f(x_0) > 0$) of x_0.

(iii) If there is no such zero, the solution converges to $-\infty$, respectively $+\infty$.

It might be that solution converges to $\pm\infty$ in finite time, not necessarily when ...

Definition

function $f : U \to \mathbb{R}$; $U \subset \mathbb{R}^{n+1}$ - open.
f is locally Lipschitz continuous if

$$L = \sup_{t,x \neq t,y \in V} \frac{|f(t,x) - f(t,y)|}{|x-y|}$$

is finite for any compact $V \subset U$. L is dependent on V.

Theorem 1.3 Let f be Lipschitz continuous with respect to x uniformly in t. Let $x(t)$ and $y(t)$ - diff. functions on $[t_0, T)$ such that $x(t_0) \leq y(t_0)$ and for all $t \in [t_0, T)$

$$\dot{x}(t) - f(t, x(t)) \leq \dot{y}(t) - f(t, y(t))$$

Then $x(t) \leq y(t)$ for all $t \in [t_0, T)$.
Shorter if $x(t) < y(t)$ for some time it remains true for all later times.

Proof. (by contradiction argument)

Let $\exists\, t_1 : x(t_1) = y(t_1)$ and $x(t) > y(t)$ for $t \in (t_1, t_1 + \varepsilon)$.

Denote $\Delta(t) = x(t) - y(t) > 0$, $t \in (t_1, t_1 + \varepsilon)$ and consider

$$\dot{\Delta}(t) = \dot{x}(t) - \dot{y}(t) \leq f(t, x(t)) - f(t, y(t)) \leq$$

$$\leq L(x(t) - y(t)) = L\Delta(t); \; (x(t) > y(t))$$

$$\leq |f(t,x) - f(t,y)| \leq L\,|x(t) - y(t)|$$

$$(\dot{\Delta}(t) - L\Delta(t) \leq 0)$$

Then $\tilde{\Delta}(t) = \Delta(t) e^{-Lt}$ satisfies $(\times e^{-Lt})$

$$\dot{\tilde{\Delta}}(t) \leq 0; \; \tilde{\Delta}(t) \leq \tilde{\Delta}(t_1) = 0 \Rightarrow$$

$x(t) \leq y(t)$ for $t \in (t_1, t+\varepsilon)$ - contradiction

$\boxed{\text{Consequence}}$ Consider $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ 6.a

if $x(t_0) \le y(t_0)$

for two solutions of the ODE above.

$$\dot{x} - f(t,x) = \dot{y} - f(t,y(t)) = 0$$

(instead of inequality $\le$)

then $x(t) \le y(t)$ for all $t \geq t_0$ when $x(t)$, $y(t)$ are defined.

If $x(t_0) = y(t_0)$ we apply the theorem two times: one time for

$$\dot{x} - f(t,x) \le \dot{y} - f(t,y(t)), \quad x(t_0) \le y(t_0)$$

and one time for

$$\dot{x} - f(t,x) \geq \dot{y} - f(t,y(t)), \quad x(t_0) \geq y(t_0)$$

and conclude that

$x(t) \le y(t)$ and $x(t) \geq y(t)$

at the same time $\Rightarrow$ therefore

$\underline{x(t) = y(t)}$ – $\boxed{\text{solution to IVP is unique!}}$

<u>Uniqueness of solutions to IVP for Lipschitz $f(t,x)$!</u>

Examples of equations that can be solved analytically (*)

1. Linear firstorder scalar equation.

$$\dot{x} + p(t)x = q(t)$$

Method of integrating factor: multiply the equation by $e^{\mathbb{P}(t)}$ with

$\mathbb{P}(t) = \int p(t)dt$. Observe that

$$\frac{d}{dt}\left(x e^{\mathbb{P}(t)}\right) = q(t)e^{\mathbb{P}(t)} \Rightarrow$$

$$x \cdot e^{\mathbb{P}(t)} = \int_0^t q(s)e^{\mathbb{P}(s)}ds + x(0)$$

$$x = x(0)e^{-\mathbb{P}(t)} + e^{-\mathbb{P}(t)}\int_0^t q(s)e^{\mathbb{P}(s)}ds$$

Equations with separable variables.

$$\dot{x} = \frac{dx}{dt} = f(x) \cdot g(t)$$

$$\underbrace{\int \frac{dx}{f(x)}}_{F(x)} = \underbrace{\int g(t)\,dt}_{G(t)} + \text{const}$$

$$F(x) = G(t) + \text{const}$$

gives an implicit connection between x and t. In particular for equation $\dot{x} = f(x)$ it gives

$$F(x) = t + \text{const}$$

Exercises.

Problem 1.1 Falling stone (of mass m) dropped from the height h.

(1) $\ddot{r} = -g$ $\quad (m\ddot{r} = -mg)$ (Newtons 2 law)

(2) $m\ddot{r} = -\dfrac{\gamma M m}{(R+r)^2}$; $g = \gamma M / R^2$

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(1) $r = -\dfrac{gt^2}{2} + at + b$

$r(0) = h$; $r'(0) = 0$; $\boxed{b = h.}$

$-gt + a \Big|_{t=0} = 0$ $\quad \boxed{a = 0}$

$r(T) = 0$

$-\dfrac{gT^2}{2} + h = 0$; $\boxed{T^2 = \dfrac{2h}{g}}$

T time to the ground.

(2) In exact equation the right hand side is smaller in absolute value, so it will be smaller velocity and larger time to reach the ground.

Problem 1.14. Charging a capacitor

$$R\dot{Q}(t) + \frac{1}{C} Q(t) = V_0$$

$Q(t)$ - charge, C - capacitance, V_0 - voltage

R - resistance of the wire. $Q(0) = 0$; $Q(t) =$

$$\dot{Q} = \left(\frac{V_0}{R}\right) - \frac{1}{RC} Q$$

$$\frac{dQ}{(V_0/R) - Q(1/RC)} = dt \; ; \quad CR \frac{dQ}{(CV_0 - Q)} = dt$$

$$-RC \ln |cV_0 - Q| = t + \varkappa$$

$$-\frac{\varkappa}{RC} = \ln(|cV_0|) \quad \text{- by initial condition } Q(0) = 0$$

$$\varkappa = -RC \ln(|cV_0|)$$

For $cV_0 > 0$, $Q(t) < cV_0$

$$\ln(cV_0 - Q(t)) = -\frac{1}{Rc}(t + \varkappa)$$

$$Q(t) = cV_0 - \exp\left\{-\frac{t+\varkappa}{Rc}\right\}$$

For $cV_0 < 0 \Rightarrow Q(t) > cV_0$

$$\ln(Q(t) - cV_0) = -\frac{1}{Rc}(t + \varkappa)$$

$$Q(t) = cV_0 + \exp\left\{-\frac{t+\varkappa}{Rc}\right\}$$

$Q(t) \to cV_0$ if $t \to +\infty$

$cV_0 > 0$ $Q(t)$ tends to V_0 from below,

Growth of bacteria (logistic equatio

Problem 1.15

(Pierre François Verhulst) ($N > 0$)
1845. - population dynamics. number of species.

$$N' = k\left(1 - \frac{N}{N_{max}}\right)N \ ; \quad n = \frac{N}{N_{max}} \ ;$$

$$n' = k(1-n)n$$ – non-dimensional form of equation.

$$\frac{dn}{n(1-n)} = k\,dt \ ; \quad \int dn\left(\frac{1}{n} + \frac{1}{1-n}\right) = \int k\,dt$$

$$\ln c + \ln|n| + \ln\left(\left|\frac{1}{1-n}\right|\right) = kt \ ;$$

consider case for $0 < n < 1$! $0 < n(0) < 1$

$$\ln\left(\frac{nc}{1-n}\right) = kt \ ; \quad \frac{nc}{1-n} = e^{kt}$$

$$cn = e^{kt}(1-n) \ ; \quad n\left(c + e^{kt}\right) = e^{kt}$$

$$n = \frac{e^{kt}}{c + e^{kt}} \ ; \quad n(0) = \frac{1}{c+1} \ ; \quad C = \frac{1-n_0}{n_0} > 0$$

$$\underset{t \to +\infty}{n(t) \to 1} \ ; \quad \underset{t \to \infty}{N(t) \to N_{max}}$$

non realistic.

$\boxed{\text{Consider other cases yourself:}}$ $n(0) < 0$; $n(0) > 1$;

Term kN describes the population growth. Term $-\frac{N}{N_{max}} \cdot N$ describes competition for food within population

Problem 1.16 Optimal harvest on logistic growth population. a)

$$\dot{N}(t) = r\left(1 - \frac{N(t)}{N_{max}}\right)N(t) - H \,; \quad N(0) = N_0 \,, \; H > 0$$

Introduce non-dimentional form of equation.

$$x(t) = \frac{N(t)}{N_{max}} \,; \quad \tau = rt$$

$$\dot{x}(\tau) = (1 - x(\tau))\cdot x(\tau) - h \,; \quad h = \frac{H}{r N_{max}}$$

$$f(x) = (1-x)x - h \qquad h > 0$$

$$\int \frac{dx}{x(1-x) - h} = \int d\tau$$

– integration depends on the number of roots to $-f(x) = x^2 - x + h = 0$

$$x_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - h}$$

$h < \frac{1}{4}$ -> two roots that are stationary points to $\dot{x}(\tau) = f(x)$; $x_1 < x_2$

$h = \frac{1}{4}$ – one double root $x_{1,2} = \frac{1}{2}$

$h > \frac{1}{4}$ – no real roots.

One can guess that if $h < \frac{1}{4}$, $x(t) \to x_2$ for $x(0) > x_1$; and $x(t) \to -\infty$ for $x(0) < x_1$. General statement about it is formulated as Lemma 1.1. (Problem 1.28). $f(x)$ here is a Lipschitz function, so the lemma is valid), because the solutions are unique.

On the other hand one can carry out the integration and get exact solution.

We consider the case $h < \frac{1}{4}$, $x_1 < x_2$.

$$\int \left(\frac{1}{x - x_2} - \frac{1}{x - x_1}\right)\frac{1}{(x_2 - x_1)}\,dx = \int -d\tau$$

Integration implies b)

$$\ln|x-x_2| - \ln|x-x_1| = -(x_2-x_1)\tau + c$$

Computing exponent of the left and right hand sides we get:

$$\frac{|x-x_2|}{|x-x_1|} = e^{-(x_2-x_1)\tau + c}$$

Initial conditions give value for c:

$$e^c = \left|\frac{x_0-x_2}{x_0-x_1}\right| \quad ; \quad c = \ln\left|\frac{x_0-x_2}{x_0-x_1}\right|$$

Depending on the position of x_0: $x_0 < x_1$, $x_1 < x_0 < x_2$, $x_0 > x_2$ we get different formulas for $x(\tau)$.

If $x_0 < x_1 \Rightarrow c > 0$, because $\left|\frac{x_0-x_2}{x_0-x_1}\right| > 1$

The equation for x looks as

$$\frac{x_2-x}{x_1-x} = e^{-(x_2-x_1)\tau + c}$$

$$x\left(e^{-(x_2-x_1)\tau + c} - 1\right) = x_1 e^{-(x_2-x_1)\tau + c} - x_2$$

$$x = \frac{x_1 e^{-(x_2-x_1)\tau + c} - x_2}{e^{-(x_2-x_1)\tau + c} - 1} \longrightarrow -\infty$$

$$\tau \to \frac{c}{(x_2-x_1)}$$

$c > 0$; $(x_2-x_1) > 0$

The solution $x(\tau)$ blows up in finite time when $x_1 < x_2$; $x_0 < x_1$

Other cases with various h an various positions for initial data x_0 can be done similarly.

1.6. Logistic equation with periodic harvesting.

$$\dot{x}(t) = (1-x)x - h(1-\sin(2\pi t))$$

($(1-\sin(2\pi t))$ can be replaced by a non-negative periodic $g(t)$.)

Poincaré map.

Let $\phi(t,x)$ - solution that starts at x at $t=0$.

$\boxed{P(x) = \phi(1,x)}$ (1 is a period of the non-uniform term)

We get a periodic solution $\Leftrightarrow$ x_0 - start point is a fixed point of the Poincaré map.

$\boxed{P(x_0) = x_0}$ $\quad \phi(1,x_0) = \phi(0,x_0) = x_0.$

Consider properties of P:

$\exists\, \theta(t,x) = \frac{\partial}{\partial x}\phi(t,x)$

$$\dot{\phi}(t,x) = (1-\phi(t,x))\phi(t,x) - h(1-\sin(2\pi t))$$

$\frac{\partial}{\partial x}!$ $\Rightarrow$

$$\dot{\theta}(t,x) = (1-2\phi(t,x))\theta(t,x)$$

integrate the equation. (linear equation)

$$\theta(t,x) = \exp\left(\int_0^t (1-2\phi(s,x))\,ds\right)$$

for $t=1$ $\Rightarrow$

$$P'(x) = \theta(1,x) = \exp\left(1 - 2\int_0^1 \phi(s,x)\,ds\right) > 0$$

$\Rightarrow$ P is strictly increasing!

Differentiating $P'(x)$ with respect to x again we get

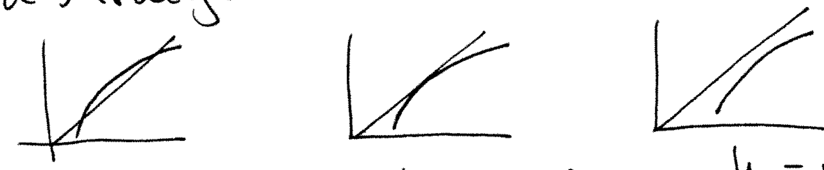
$$P''(x) = -2\left(\int_0^1 \Theta(s,x)\,ds\right) P'(x) < 0$$

because $\Theta(s,x) > 0$, $P'(x) < 0$.

Therefore $P(x)$ is strictly increasing and concave.

Therefore $P(x)$ can have maximum 2 intersection points with line $y = x$.

P' is a decreasing function and cannot cross a straight line more than two times.



- numerically

$h = 0.1$ $h = 0.249$ $h = 0.5$

We investigate P-dependence on h analytically.

$\exists$ $\Psi(x,t) = \frac{\partial}{\partial h}\Phi(t,x)$

$$\dot{\Psi}(t,x) = \left(1 - 2\Phi(t,x)\right)\Psi(t,x) + \left(1 - \sin(2\pi t)\right)$$

$$\Psi(0,x) = \frac{\partial}{\partial h}\Phi(0,x) = 0 \quad \Rightarrow \text{linear equation}$$

$$\Psi(t,x) = -\int_0^t \exp\left(\int_s^t \left(1 - 2\Phi(r,s)\right)dr\right) \times \left(1 - \sin(2\pi s)\right) ds$$

$t = 1 \Rightarrow$

$\frac{\partial}{\partial h} P_h(x) < 0$ - decreasing

For $h = 0$ $P_0(x) = \dfrac{ex}{1 + (e-1)x}$

| $x_1 = 0$; $x_2 = 1$ - fixed points! |

for some value h_c $P(x)$ looses fixed point and all orbits $\to -\infty$ because $P(x) < x$ $\forall x \in \mathbb{R}$.

Suppose $h < h_c$ and $x_1 < x_2$ - fixed points of $P(x)$.

Consider iterates $P^n(x) = P(P^{n-1}(x))$

We claim

$$\lim_{n\to\infty} P^n(x) = \begin{cases} x_2 & x > x_1 \\ x_1, & x = x_1 \\ -\infty, & x < x_1 \end{cases}$$

Since $P(x)$ is strictry increasing

$x_1 = P(x_1) < P(x) < P(x_2) = x_2$

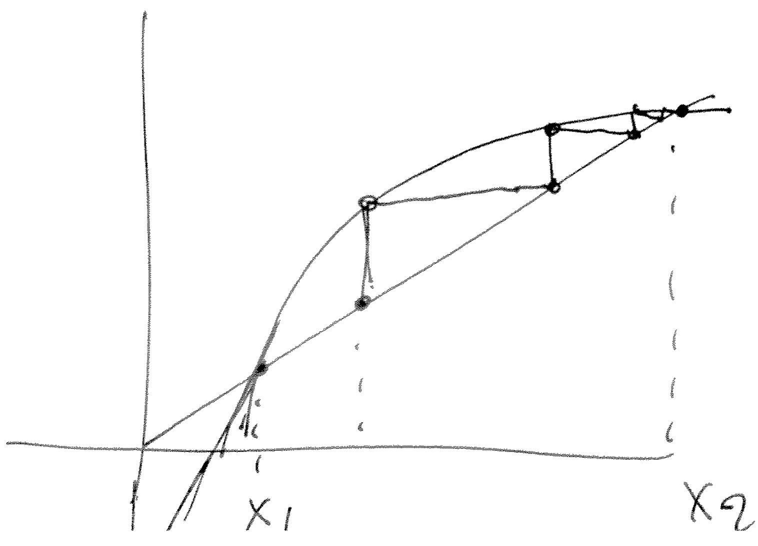
since $P(x)$ is concave $\Rightarrow x < P(x)$.

It implies that $P^n(x)$ is strictly increasin and $(< x_2)$

$\Rightarrow \exists \lim\limits_{n\to\infty} P^n(x) = x_0$.

$\Rightarrow \underline{P(x_0)} = P(\lim\limits_{n\to\infty} P^n(x)) = \lim P^{n+1}(x) = \underline{x_0}$

$\Rightarrow$ fixed point $\Rightarrow \underline{x_0 = x_2}$!



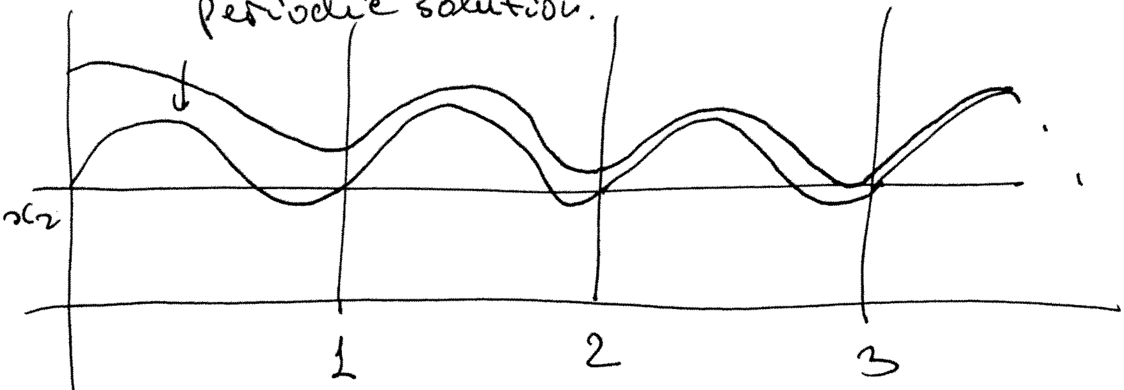
Other cases are shown similarly.

What happens with trajectories $\phi(t,x)$ when $t \to +\infty$?

a) We know that $\phi(n,x) \underset{n\to\infty}{\to} -\infty$ for $x < x_1$

b) and $\phi(n,x) \underset{n\to\infty}{\to} x_2$, $x > x_1$.

We live to conclude that even $\phi(t,x) \underset{t\to\infty}{\to} \phi(t,x_2)$



We say that $\phi(t,x_2)$ is a stable solution and is an attractor.

It is easy to observe that $x' \le x$

$\Rightarrow$ $x(t+n) \le e^{t}(x(n))$. $\boxed{\text{Take } 0 < t' < 1}$

and $\boxed{x(t'+n) - x_p(t'+n) \le e^{t'}(x(n) - x_p(n))}$

But $x(n) = \phi(n,x)$; $x_p(n) = \phi(n,x_2)$

$x(n) - x_p(n) \underset{n\to\infty}{\to} 0 \Rightarrow$

$\phi(t,x) - \phi(t,x_2) \underset{t\to\infty}{\to} 0$

Clarified solution is

on the next page

Logistic equation with periodic harvesting.

Convergence to periodic solution.

if $x(0) > x_p(0) = x_2$

then $x_p(t) < x(t) \quad \forall t$

because of uniqueness. (right hand side is locally uniformly Lipschitz)

Consider $\Delta(t) = x(t) - x_p(t)$

$t = n + t'; \quad t' \leq 1$

$$\dot{\Delta}(t) = f(x(t)) - f(x_p(t)) =$$

$$= x(t) - x_p(t) - \underbrace{\left(\frac{x^2(t)}{N_m} - \frac{x_p^2(t)}{N_m} \right)}_{\geq 0} \leq$$

$$\leq x(t) - x_p(t)$$

$$\Rightarrow \dot{\Delta}(t) \leq \Delta(t) \Rightarrow$$

$$\boxed{0 \leq \Delta(t) \leq \Delta(t_n)\, e^{t'}}$$

$\Delta(t_n) \to 0 \quad \Rightarrow \quad \Delta(t) \Rightarrow 0$

$$\Rightarrow \boxed{x(t) - x_p(t) \underset{n \to \infty}{\longrightarrow} 0}$$