List of questions and typical problems to examination in ODE and Mathematical Modeling MMG511/TMV160.
One must know (in this order!):

- **definitions to all notions,**
- **all formulations of the theorems from the list,**
- **must be able to prove theorems marked by yellow,**
- **must be able to solve problems of the types mentioned in the list.**

Topics, definitions and notions	Methods, theorems, lemmas and corollaries	Typical problems
Classification of ODE: order, autonomous, non-autonomous, linear. Initial value problem: existence, uniqueness. Integral curve. Maximal existence time for solutions of 1-dimensional autonomous ODE	Solutions to first order 1-dimensional autonomous ODE: existence, uniqueness, extensibility, limits of solutions: §1.3 p. 9, formulas 1.21- 1.25 Lemma 1.1, p. 21 on trajectories of scalar autonomous ODEs and structure of their phase portrait. Theorem 1.3 with the consequence on uniqueness, p.27	Solve 1-dimensional first order ODEs: linear ODE, ODE with separable variables. Find maximal existence time. Decide if an equation has unique solutions Give examples of non-uniquness of solutions (like pp. 10-11.) Find fixed points and limit properties of solutions. Identify Lipschitz functions.
Lipschitz continuous functions,	§ 1.5 Theorem 1.3	
§3.1-2. Linear system of ODE with constant matrix. x'=Ax Matrix norm. Change of variables. Equivalent matrices U^-1AU and A. Polynom P(A) and exponent exp(A) of a matrix A. det A, trace Tr(a) for equivalent matrices. Jordan canonical form of matrix J, f. 3.15-16 Jordan block: formula 3.16, p.61, 3.192, p.106 Block diagonal matrices. Generalized eigenspaces and generalized eigenvectors: formula 3.184, p.104 Transformation leading to Jordan canonical form of matrices: U^-1AU=J, A=UJU^(-1). (lecure notes or f. 3.192, p. 106)	For commuting matrices: AB=BA exp(A+B)=exp(A)exp(B) Lemma 3.1. p. 6. Theorem 3.2 on Jordan canonical form, p. 61 Exponent of Jordan block: 3.19, p.63, 3.42, p. 70 Polynomial and exponent of block diagonal matrices. 3.17, p.62 and lecture notes. Exponent of an arbitrary matrix using 3.17, p.62. Classification of phase portraits of linear systems with constant matrix. p. 67-70 and lecture notes. Theorem 3.4 on stable and unstable generalized eigenspaces. p.71 Corollary 3.5, on stable linear systems. p. 71 Corollary 3.6, on asymptotically stable linear systems p.71 Duhamel formula 3.48 p.72 for non homogeneous linear systems with constant matrix	Compute exponent of a 2x2 matrix or an arbitrary Jordan matrix. 2-dimensional linear systems in plane: classify and draw phase portraits (Problem 3.14). Solve IVP for a linear autonomous system in the plane or in space if eigenvalues are given or are easy to calculate . Solve inhomogeneous linear systems of ODE with constant matrix. Decide if a vector valued function can be solution to a linear system ODE. (Problem like 3.12. p 73) Use Corollaries 3.5, 3.6 to conclude about stability or asymptotic stability of the fixed point in the origin.
Non-autonomous linear systems Principle matrix solution. f. 3.81, p.81 Fundamental matrix solution. p. 83. Stable and asymptotically stable linear systems (origin is in this case stable or unstable fixed point according to the general notion of fixed point that comes later in Chapter 6.)	Theorem 3.10 on existence of solutions Superposition principle. P.81 Theorem 3.10. p.82 on the dimension of the solution space (proof in lecture notes). Duhamel formula 3.97 p.84 for non-homogeneous non- autonomous linear equations. Liouvilles formula 3.91, p.83	Find principle matrix solution for a simple equation that can be solved explicitly.
Linear systems with periodic coefficients. Monodromy matrix. f. (3.119), p. 91 Floquet exponents. P. 93 Floquet multipliers. Matrix logarithm 3.121, p. 91, 3.200, p. 108	Principal matrix of periodic linear system is is periodic: Lemma 3.14, p.91 Floquet theorem 3.15, p.92. Formula 3.125 on the structure of principal matrix for a linear system x'=A(t)x with periodic matrix A(t). Corollary 3.17, p. 93 about the stability of periodic linear systems.	Find a monodromy matrix for a simple equation that can be solved explicitly.
§2.1 Fixed point theorems Normed vector space, p. 33 Cauchy sequence 2.5, p.34 complete space, Banach space: p.34. Space C(I) of continuous functions on a compact I. Uniform convergence p.34 Open, closed, compact, connected sets, Fixed point of an operator. p.35 Contraction. p.35 Sequence of iterations, p. 34 in the proof of Theorem 2.1	Banach contraction principle. Theorem 2.1 p.35 Theorem 2.4, p. 39. Method with telescoping sums in the proof to Theorem 2.1	Show that an operator is a contraction in C(I). Show using Banach's contraction principle that an operator has a fixed point in a ball.

Topics, notions, definitions	Methods, theorems, lemmas and corollaries	Typical problems
§2.2, Existence and uniqueness theory for IVP (Initial Value Problem) Integral formulation of IVP. 2.11, p. 36 Picard iterations. 2.13, p. 36 Lipschitz functions. 2.18 , p. 37	Picard-Lindelöf theorem. 2.2 Two variants of the proof: by contraction principle p. 37-38, and by an exponential estimate (in lecture notes and in the proof to Theorem 2.5 that we did not study)	Identify Lipschitz functions of several variables. Write explicitly 2-3 Picard iterations for an equation.
§2.4 Dependence of solutions on initial data and right hand side Gronwall's inequality 2.38, p. 43. (only this simpler variant) Well posed problems: dependence of initial data, dependence on right hand side, p.42.	Gronwall's inequality, 2.38, p. 43: proof in lecture notes. Theorem 2.8, formula 2.40 p. 43 on the dependence of solutions on initial data and on the right hand side of ODE	Use Theorem 2.8 to estimate difference between solutions a particular ODE with different initial data or between solutions to two particular ODE
§2.5 Extensibility of solutions Compact set. Extensibility Maximal existence interval (T-,T+), maximal solution.p.51 Global solution p. 51	Lemma 2.14, p. 52 on extensibility of solutions for which values x(t_m) on a time sequence of t_m ->T+ converge to a point y in the domain U of the right hand side. Corollary 2.15 on extensibility of solutions that stay within a compact subset of the domain U of the right hand side. Corollary 2.16 on non extensibility of solutions p. 53 : solution leaves any compact if the maximal existence time T+ is bounded. Global existence for equations with linear growth of the right hand side. Theorem 2.17 p. 53	Decide for solutions, starting in a certain domain how long they can be extended and which limits they might have.
§3.7 Stability by linearization. Linearization of ODE Definitions of stable and exponentially stable fixed points in nonlinear case comes in Chapter 6, p.198, but Theorem 3.26 and Corollary 3.27 deal in fact with asymptotic stability of the fixed point in the origin.	Stability of the perturbation of a linear non – autonomous ODE by a non-linear right hand side. Theorem 3.26, p. 101. Stablity of autonomous non-linear ODE by linearization. Corollary 3.27. p.101 Proofs of both statements use Gronwall's inequality and are the same as to Theorem 3.20, a proof is also given in lecture notes.	Use stability theorems to decide under which conditions a solution starting at some distance to the origin goes to zero as t goes to infinity.
Autonomous systems. **Stability of fixed points by linearization** §6.2-6.3 maximal integral curves, orbits, fixed points(orbits), periodic orbits, non-periodic orbits, invariant sets, limit sets.pp.192-193 Stable fixed points. Asymptotically stable fixed points. p. 198, Unstable fixed points (check an explicit definition in lecture notes)	Asymptotic stablity of autonomous non-linear ODEs by linearization Theorem 6.10 (the same as Corollary 3.27 p. 101)	Prove that an ODE has a positively invariant set. Show stability of a fixed point using Theorem 6.10.
Stability of fixed points by Liapunovs functions Liapunovs function, p. 200 strict Ljapunov function, p.201 Derivative (Lie derivative) of a scalar function along trajectories. 6.41, p. 202 More practical definitions for smooth C^1 Liapunovs functions 6.40, p.202 and in lecture notes.	Stability of fixed points to autonomous ODE by Liapunovs functions: Theorem 6.13. (simple proof in lecture notes, a more involved one in Lemmas 6.11, 6.12 p.201) Asymptotic stability of fixed points to autonomous ODE by strict Liapunovs functions (follows from Theorem 6.14 and from a separate proof in lecture notes) Asymptotic stability of fixed points to autonomous ODE by "weak" Liapunovs functions and Krasovskii-LaSalle Theorem 6.14., p.202 Instability of a fixed point by Liapunov's method (lecture notes)	Find a Liapunov function (strict Liapunovs function) Show stability (asymptotic stability) of a fixed point of an ODE using theorem 6.13 or a strict Liapunov function. Apply Krasovskii-LaSalle Theorem 6.1 to show asymptotic stability of a fixed point using a "weak"Liapunov function Show instability of a fixed point.
Examples of periodic solutions to autonomous systems in plane §6.4 The Poincare map. F. 6.25 P.197 Ljenard and Van der Pol equations	Connection between Poincare map and periodic solutions, bottom of p.197 Phase portrait for Volterra Lotka equation p. 210 Poincare Bendixson theorem 6.7.13 p. 222	Prove that an ODE has at least one periodic solution by Poincare Bendixson theorem. Prove that an ODE in plane

Volterra Lotka equation, p. 209	A limit set of a solution in a compact positively invariant set without fixed points is a periodic orbit. (without proof). Problem 7.11, (Bendixson's criterion) p. 227	does not have periodic solutions using Bendixsons criterion.