Stability of fixed points to autonomous systems of ODE and related notions.

(after parts of Chapter 6 with some additions and clarifications)

We consider an autonomous ODE

$$\begin{cases} \dot{x}(t) = f(x) \\ x(0) = x_0 \end{cases} \qquad \underline{\text{I.V.P.}}$$

where $f \in C^1(M, \mathbb{R}^n)$, $M \subset \mathbb{R}^n$, M - open set.

Initial time in autonomous systems can always be set $t_0 = 0$ because the right hand side f is independent of time. f can be interpreted as a vector field in $\mathbb{R}^n$ an can be illustrated by vectors in plane if the system is 2-dimensional by command quiver(X,Y,U,V) in Matlab. X, Y - are arrays for points coordinates, U, V - are arrays for vectors coordinates.

Definition

Solutions $\varphi(t)$ to I.V.P are called <u>integral curves</u> or <u>trajectories</u>.

One can always consider a <u>maximal integral curve</u> for every point x

defined on a maximal interval $I_x = (T_-(x), T_+(x))$ and notated as $\varphi_x(t)$.

When we are interested not time evolution of the system but only in the path in M that the solution $\varphi_x(t)$ goes through, it is useful to consider orbits of points x in M.

Definition

If $x \in M$ is a „starting point" (x_0 in previous notations) then the image of the maximal solution on the interval $(T_-(x), T_+(x)) = I_x$ is called orbit of x and is denoted by

$\gamma(x) = \varphi_x(I_x)$

It consists of all values that the solution with initial point x attain for times from $I_x = (T_-(x), T_+(x))$

Similarly one can consider forward and backward orbits

$$\gamma_+(x) = \varphi_x([0, T_+(x))), \quad \gamma_-(x) = \varphi_x((T_-(x), 0])$$

There are some types of orbits of particular interest as:

i) fixed orbits (fixed points)

ii) periodic orbits, corresponding to periodic solutions with the period T being a minimal $T > 0$ such that $\varphi_x(t+T) = \varphi_x(t)$, $t, t+T \in I_x$

iii) non-closed orbits (infinite orbits, orbits with an endpoint on the boundary of M, orbits connecting two fixed points.

Definition

A set $C \subset M$ is called + or − invariant or σ-invariant ($\sigma = \pm$) if $\gamma_+(x)$, correspondingly $\gamma_-(x) \in U$ for all $x \in U$.

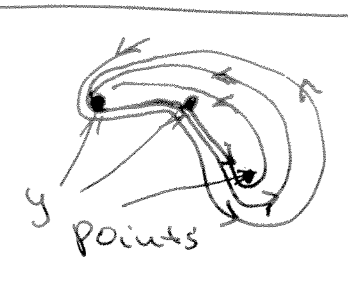
+ invariant sets are also called positively invariant sets.

Definition The ω_+ (ω_-) limit set of a point $x \in M$ is the set of those points $y \in M$ that $\exists \{t_n\}_{n=1}^{\infty}$ $t_n \xrightarrow[n\to\infty]{} \pm\infty$ such that $\varphi_x(t_n) \xrightarrow[n\to\infty]{} y$

For $\omega_\pm(x)$ to be not empty T_+ (T_-) must be $+\infty$, ($-\infty$).

The meaning is that with time growing (diminishing) the solution comes arbitrarily close to each point $y \in \omega_+(x)$ ($\omega_-(x)$) at some t_n - moments of time.

Definition



The point x is called + (−) complete if $T_+(x) = +\infty$ ($T_-(x) = -\infty$)

An important property of limit sets $\omega_\pm(x)$ is that

Lemma 6.5 The sets $\omega_\pm(x)$ are closed invariant sets.

Lemma 6.6 - If $\gamma_\pm(x) \subset C$ - compact then $\omega_\pm(x)$ is non-empty, compact and connected.

We leave these results without proof.

Stability of fixed points to autonomous systems.

Consider $\dot{x} = f(x)$; $x_0 \in M$, $M \subset R^n$, M-open; $f \in C^1(M, R^n)$
$f(x_0) = 0$, x_0 - fixed point.

Definition fixed point x_0 is stable if for any neighborhood $U(x_0)$ there is another neighborhood $V(x_0) \subset U(x_0)$ such that $\forall x \in V(x_0)$ positive orbit starting in x: $\gamma_+(x) \subset U(x_0)$, or by other words a trajectory $\varphi_x(t)$ starting in $V(x_0)$ will stay in $U(x_0)$ for all $t \geq 0$.

A fixed point x_0 is called unstable if it is not stable.

Definition a fixed point x_0 is called asymptotically stable if it is stable and in addition to that $\exists$ a neighborhood $U_a(x_0)$ such that $\forall x \in U_a(x_0)$

$$\lim_{t \to +\infty} |\varphi(t,x) - x_0| = 0.$$

The last limit property does not imply usual stability, so the first requirement in the definition is necessary. The next counterexample shows it.

Problem 6.16. Consider the system

$$\dot{x} = x - y - x(x^2+y^2) + \frac{xy}{\sqrt{x^2+y^2}}$$

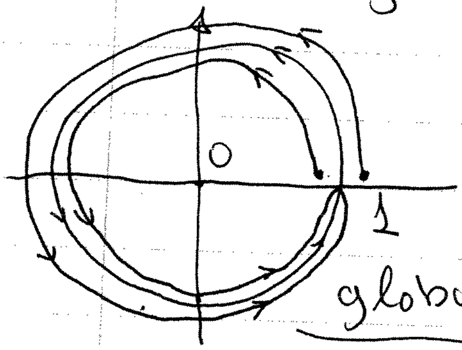
$$\dot{y} = x + y - y(x^2+y^2) - \frac{x^2}{\sqrt{x^2+y^2}}$$

The fixed point $(1,0) = x_0$ is not stable despite the fact that $\lim_{t \to +\infty} |\varphi(t,x) - x_0| = 0$ for any solution $\varphi(t,x)$ starting in a small neighborhood of $(1,0) = x_0$. One can observe these properties by analysing the system in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. One gets equations for r and θ by differentiating $r^2 = x^2 + y^2$ and $\tan(\theta) = \frac{y}{x}$ and using equations for $\dot{x}$ and $\dot{y}$ above.

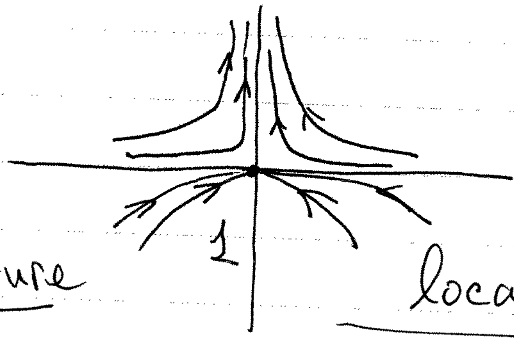
It leads to equations for $r(t)$ and $\theta(t)$

$$\begin{cases} \dot{r} = (1-r^2)r \\ \dot{\theta} = 2\left(\sin\left(\frac{\theta}{2}\right)\right)^2 \end{cases}$$

Considering the firs equation it is easy to observe that trajectories starting from $1 < r < r+\varepsilon$ tend to $r=1$ (from above), and trajectories starting at $1-\varepsilon < r < 1$ tend to $r=1$ from below. Trajectories starting at $r=1$ stay on the circle $r=1$.



global picture



local picture

$\theta(t)$ is non-decreasing function of t

$\left(\sin\frac{\theta}{2}\right)^2 \geq 0$ and $\sin\left(\frac{\theta}{2}\right) = 0$ for $\theta = 0$, $\theta = 2\pi$, otherwise $\sin\left(\frac{\theta}{2}\right) > 0$ $0 < \theta < 2\pi$

It makes that trajectories starting close to $(1,0)$ at $x = (1+\delta, \varepsilon)$, $\varepsilon > 0$, $|\delta|$-small go around the circle $r=1$ and tend to $(1,0)$ "from below" the x-axis.

These trajectories are <u>unstable:</u> they leave any neighborhood of $(1,0)$ and go to the limit only after going around the origin, so $\varphi(t,x) \xrightarrow[t\to\infty]{} x_0$

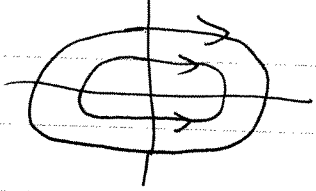
Another example shows that a stable fixed point can be not asymptotically stable.

$\dot{x}_1 = x_2$; $\dot{x}_2 = -x_1^3$

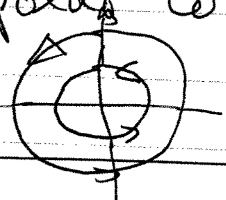
using chain rule

Excluding time we get an equation for orbits of this system.

$$\frac{dx_2}{dx_1} = \frac{-x_1^3}{x_2}; \qquad \boxed{\frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 = \text{const}}$$



A simpler similar example is the linear system $\dot{x}_1 = +x_2$, $\dot{x}_2 = -x_1$ with orbits - closed circles and equations in polar coordinates: $\dot{r} = 0$ $\dot{\theta} = 1$;



In both cases trajectories starting close to the origin stay close to the origin, but they do not tend to the origin with $t \to +\infty$.

Definition Teschl introduces even a stronger notion - an exponential stability for a fixed point x_0, that means $\exists\, \alpha, \delta, C > 0$ constants such that

$$|\Phi(t,x) - x_0| \leq C e^{-\alpha t}|x - x_0|,$$

$$|x - x_0| \leq \delta$$

It implies stability and asymptotic stability.

Definition

If the inequality in the definition is strict : $L(\phi_x(t_1)) < L(\phi_x(t_0))$, $x \in U(x_0)$, $x \neq x_0$.
the function L is called <u>strict</u> Liapunov's function.

There is a variant of the definition that easily to check. If we take $L \in C^1(U(x_0), R)$, then monotonicit can be expressed in terms of the derivaty along trajectories.
L is nonincreasing along trajectories is $L(\varphi_x(t))' = \nabla L \cdot \dot{\phi} = \nabla L \cdot f \leq 0$
L is strictly decreasing along trajectories if $L(\phi_x(t))' = \nabla L \cdot \dot{\phi}_x(t) = \nabla L \cdot f(x) < 0$ for all $x \in U(x_0)$, $x \neq x_0$.

It gives simpler criteria for Liapunov's and strict Liapunov's functions.

Theorem 6.13 (Liapunov) Suppose x_0 be a fixed point of $\dot{x} = f(x)$
If there is a Liapunov's function L, then x_0 is stable.

Proof in the book is splitted in two lemmas, a bit too complicated. We give a simpler proof.

Theorem 6.10

Exponential stability by linearization

Suppose $f \in C^1$ has a fixed point x_0 and suppose that all eigenvalues of the Jacobi matrix at x_0 have negative real part. Then x_0 is exponentially stable.

This theorem is the same as Corollary 3.27. before.

Theorem 6.10 cannot be applied in cases when Jacobi matrix is degenerate or has purely imaginary eigenvalues. In this case a more refined information about the system can help.

Definition of Liapunov functions (1892)

Let $L : U(x_0) \to R$ - continuous in a neighborhood $U(x_0)$ of x_0.

1) $L(x_0) = 0$
2) $L(x) > 0, \quad x \neq x_0, \quad x \in U(x_0)$
3) if $t_1 > t_0$, $\phi_x(t_1) \in U(x_0)$, $\phi_x(t_0) \in U(x_0)$ then $L(\phi_x(t_1)) \leq L(\phi_x(t_0))$, $x \in U(x_0)$

L is called Liapunov's function.

Liapunov's functions is not increasing along trajectories of the differential equation $\dot{x} = f(x)$.

Proof of Th. 6.13 (after Hirsh and Smale)
Let $\delta > 0$ be so small that the ball $\overline{B_\delta(x_0)} \subset U(x_0)$. Let α be minimum of L-function on the spere $S_\delta(x_0)$ - boundary of the ball $B_\delta(x_0)$.

Then $\alpha > 0$ because $L(x) > 0$ outside x_0. The minimum exists because L is continuous and $S_\delta(x_0)$ is a compact set.

$$\text{Let } U_1 = \{ x \in \overline{B_\delta(x_0)} \mid L(x) < \alpha \}$$

U_1 is an open set as inverse map of the open set $(-\infty, \alpha)$. $U_1 \ni x_0$ and is a neighborhood of x_0. No one trajectory $\Phi_x(t)$ starting inside U_1, $x \in U_1$ can meet $S_\delta(x_0)$ because L is non increasing on trajectories and $\min_{x \in S_\delta(x_0)} L(x) = \alpha$. Hence $\Phi_x(t)$ with $x \in U_1$ never leaves $B_\delta(x_0)$. ▨

An advantage of Liapunov functions in comparison with the method with linearization is that Liapunov functions give an estimate for stability domain. Linearization works only in a small neighborhood of the fixed point.

Example of Liapunov's function.

$x' = -3x^3 - y$ origin is a fixed point

$y' = x^5 - 2y^3$

We try a function $L(x,y)$ in the form $L(x,y) = ax^{2m} + by^{2n}$, $a, b > 0$

$$\frac{d}{dt}\Big(L(x(t), y(t))\Big) = 2max^{2m-1} \cdot \dot{x} + 2nby^{2n-1}\dot{y}$$

$$= 2ma\, x^{2m-1}(-3x^3 - y) + 2nby^{2n-1}(x^5 - 2y^3)$$

$$= -6max^{2m+2} - 2max^{2m-1}y + 2nby^{2n-1}x^5 - 4nby^{2n+2}$$

Let $2m-1 = 5$, $m = 3$

$2n-1 = 1$, $n = 1$

$$\Rightarrow \frac{d}{dt}L = -18ax^8 - \underbrace{6ax^5y + 2byx^5}_{} - 4nby^{2n+2}$$

take $a = 1$; $b = 3$

$L(x,y) = x^6 + 3y^2 > 0$, $(x,y) \neq (0,0)$

Liapunov's function (even strict) $\Rightarrow (0,0)$ is an asymptotically stable fixed point.

Theorem 6.14 Krasovski - La Salle principle.

$\exists$ x_0 - fixed point. $\exists$ L be a Liapunov's function which is not constant on any orbit lying entirely in $\bar{V}(x_0)\setminus\{x_0\}$.

Then x_0 is asymptotically stable.

(valid in particular in case of L - strict Liapunov function)

Moreover any trajectory lying entirely in $\bar{V}(x_0)$ converges to x_0.

Proof. $\forall x$ with $\varphi(t,x) \in \bar{V}(x_0)$, $t \geq 0$

$\exists$ by monotonicity $\lim\limits_{t\to+\infty} L(\phi(t,x)) = L_0(x)$

$\forall y \in \omega_+(x)$ $\exists \{t_n\}$; $t_n \to \infty$: $\phi(t_n,x) \to y$

$L(y) = \lim\limits_{n\to\infty} L(\phi(t_n,x)) = L_0(x)$ - const(x)

$\omega^+(x)$ is a positively invariant set by Lemma 6.5

Therefore if L is not constant on any orbit in $\bar{V}(x_0)\setminus\{x_0\}$ $\Rightarrow$ it must be $\omega_+(x) = \{x_0\}$

It holds for any $x \in S'_\varepsilon$ $\Rightarrow$

for small (enough) $\varepsilon > 0$ x_0 is asymptotically stable.

A key observation in the proof is that $L(y)$ is the same number $L_0(x)$ for all $\omega_+(x)$ limit points of the point x.

$\omega_+(x)$ consists of orbits of the system.

Example of application of the Krasovskii - LaSalle theorem.

It might be difficult to find a strict Liapunov's function and be an easier task to find a „weak" one.

Consider system.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - (1 - x_1^2)\, x_2$$

$x_0 = (0,0)$ is a fixed point.

$L(x_1, x_2) = x_1^2 + x_2^2$ is a „weak" Liapunov's function.

$$\left(L(x_1(t), x_2(t))\right)' = -2x_2^2(1 - x_1^2) \leq 0$$
for $|x| < 1$

It implies that the origin is a stable fixed point.

We observe that $L(x_1, x_2)$ vanishes on the line $x_2 = 0$ and $x_1 = \pm 1$. The last two points are „far" from the origin x_0 and are out of interest.

However no whole trajectories lie in the line $x_2 = 0$, because if $x_2 = 0$ $x_2' = -x_1 \neq 0$ and all trajectories cross the line $x_2 = 0$ (outside (0,0)). and do not lie on this line.

Theorem on instability (absent in the book)

Let $\dot{x} = f(x)$, $f(x) \in C^1(M, \mathbb{R}^n)$, $M \subset \mathbb{R}^n$, M-open. $x_0 \in M$, $f(x_0) = 0$.

1) Let $\overline{B_\delta(x_0)} \subset M$; $L \in C^1(D(\ ), \mathbb{R}^1)$
$\overline{B_\delta(x_0)} \in D(L) \subset M$.

2) $\exists \{x_n\}_{n=1}^{\infty}$, $x_n \in \overline{B_\delta(x_0)}$; $x_n \xrightarrow[n\to\infty]{} x_0$
$L(x_n) > 0$

3) $(\nabla L \cdot f)(x) > 0$ for $x \neq x_0$, $x \in D(L)$

4) $L(x_0) = 0$.

Then x_0 is unstable.

Proof. We are going to show that for any x_* with $L(x_*) > 0$ $\phi_{x_*}(t)$ leaves $\overline{B_\delta(x_0)}$.

$$(\nabla L \cdot f)(\phi_{x_*}(t)) = (L(\phi_{x_*}(t)))' > 0$$

Therefore $L(\phi_{x_*}(t))$ is an increasing function $L(\phi_{x_*}(t)) > L(x_*) > 0$

L is continuous, therefore $\exists \varepsilon > 0$ such that $0 \le L(x) < \frac{1}{2} L(x_*)$ for $|x - x_0| < \varepsilon$ $(x \in B_\varepsilon(x_0))$

Therefore the trajectory $\phi_{x_*}(t)$ stays outside the ball $B_\varepsilon(x_0)$.

The function $(\nabla L \cdot f)(x)$ is also continuous and must attain its minimum K on the compact set $B_\delta(x_0) \setminus B_\varepsilon(x_0)$:

Therefore $\left(L(\phi_{x_*}(t))\right)' \geq K > 0$

for all t such that $\phi_{x_*}(t) \in B_\delta(x_0)$

and

$$L(\phi_{x_*}(t)) \geqslant L(x_*) + Kt.$$

It implies that L must attain arbitrarily large values on the compact set $\overline{B_\delta(x_0)}$ that is impossible because L is continuous. Therefore the trajectory $\phi_{x_*}(t)$ must leave the ball $B_\delta(x_0)$ at some time.

We have shown it for arbitrary x_n with $L(x_n) > 0$ and it implies that for points x_n arbitrarily close to the fixed point x_0 trajectories starting in x_n leave the Ball $B_\delta(x_0)$. Therefore x_0 is an unstable fixed point. $\blacksquare$

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Example Show that the origin is an unstable fixed point for the system $\dot{x}_1 = x_1^2$; $\dot{x}_2 = 2x_2^2 - x_1 x_2$

With help of function

$$L(x_1, x_2) = \alpha x_1^3 + \beta x_1^2 x_2 + \gamma x_1 x_2^2 + \delta x_2^3$$

$$\dot{L}(x_1(t), x_2(t)) = 3\alpha x_1^4 + \beta x_1^3 x_2 + (2\beta - \gamma) x_1^2 x_2^2 + (4\gamma - 3\delta) x_1 x_2^3 + 6\delta x_2^4$$

Take $\alpha = \frac{1}{3}$; $\beta = 4$; $\gamma = 2$; $\delta = \frac{4}{3}$

$$\Rightarrow \dot{L}(x_1(t), x_2(t)) = x_1^4 + 4x_1^3 x_2 + 6x_1^2 x_2^2 + 4x_1 x_2^3 + 8x_2^4$$
$$= (x_1 + x_2)^4 + 7x_2^4 > 0 \qquad (x_1, x_2) \neq (0,0)$$

$$L = \frac{1}{3}x_1^3 + 4x_1^2 x_2 + 2x_1 x_2^2 + \frac{4}{3}x_2^3$$

$L(\,) = \frac{1}{3}x_1^3$, for $x_2 = 0$ and positive on the positive axis x_1.

The positive axis x_1 contains points arbitrarily close to the origin.