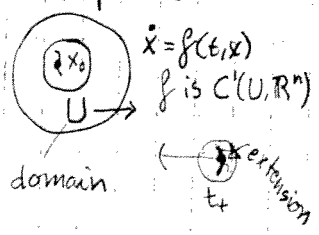
① <u>Existence</u> is used for:

<u>Extensebility</u>

one step more



$\dot{x} = f(t,x)$

f is $C^1(U, \mathbb{R}^n)$

② <u>Uniqueness</u> is used for

$\boxed{\dot{x} = Ax}$

Uniqueness is used to show the space of solutions is n-dimensional

Floquet theory

$\dot{x} = A(t)x$

<u>A(t) periodic</u>

$\Pi(t+T, t_0+T) = \Pi(t, t_0)$

Uniqueness follows from:

I: Contraction map theorem by Banach used to get solution as a unique fixed point.

II. Wellposedness (dependence on initial data and righ hand side.

$\dot{x} = f(t,x)$ $\quad M = \sup|f-g|$

$\dot{y} = g(t,y)$ $\quad L = Lip(f)$

$|x(t) - y(t)| \le |x_0 - y_0| e^{L(t-t_0)}$

$C^1 \Rightarrow$ Lipschits konst is sup. of derivative

Lipschits constants for $f \in C^1$:

$= \sup_{(t,x)\in V} \left\| \frac{\partial f}{\partial x}(t,x) \right\|$ matrix norm

<u>How to find Lipschitz constant</u>

$|f(x) - f(y)| \le f'(c)|x-y|$

intermidiate point

Why the dimension of the solutionspace is n?

A $n \times n$-matrix

⊗ $\dot{x} = A(t)x$ 1. $\{x_i(t)\}$ are solutions then $\sum_{i=1}^{0} a_i \; x_i(t)$ is a solution

Solutions to ⊗ build linear vectorspace.

2. The space of initial data is $\mathbb{R}^n$ - n-dimensional

<u>why?</u>

$x(t)$

$\underline{\Pi(t,t_0) \to \mathbb{R}^n} \to ?$

$\Pi(t_0, t_0) = I$;

$\det \Pi(t,t_0) \neq 0$

$\det \Pi(t,t_0) = e^{\int_{t_0}^{t} Tr(A)\,ds} \neq 0$

$\Rightarrow \Pi(t,t_0)$ is a bijection (keeps the dimension)

Simple argument based on the uniqueness property

$\sum_{i=1}^{n} a_i \, x_{0i} \neq 0$, a_i not all zeros

x_{0i} - linearly independent

Consider corresponding solution $\sum_{i=1}^{n} a_i \, x_i(t)$

$\dot{x}_i = A(t)x_i$

$x_i(t_0) = x_{0i}$

Argue by contradiction:

Suppose there are $\{a_i\}_{i=1}^{n}$ not all zeros such that $\sum a_i x_{0i} \neq 0$ and $\sum a_i x_i(t) = 0$ namely that $x_i(t)$ become linearly dependent at some time $t > t_0$

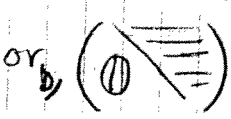
But it is impossible because the only (unique) solution that is $x(t) = 0$ at some time t_x is $x(t) \equiv 0$

Calculate principle matrix solution $\Pi(t,t_0)$ for some simple cases.

If A-constant $\Rightarrow$ $\boxed{\Pi(t,t_0) = e^{A(t-t_0)}}$ $\quad \dot{x} = Ax$

$\dot{x} = A(t)x$ $\quad$ when one can calculate $\Pi(t,t_0)$ on the paper (write down an analytical solution)

$\Pi = [\pi_1 | \pi_2 | \cdots]$

$\dot{\pi}_i = A(t)\pi_i$ $\quad$ 1. $A(t)$ is triangular a) or b) 

$\pi_i(t_0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$

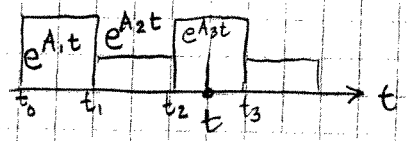
a) We can eliminate unknowns one by one solving the scalar first order ODE each step.

$$\dot{x} + p(t)x = q(t)$$

$$P(t) = \int_{t_0}^{t} p(s)\,ds$$

$$x(t) = x_0 e^{-P(t)} + e^{-P(t)} \int_{t_0}^{t} q(s) e^{P(s)}\,ds$$

2. One can get an explicit solution of $\dot{x} = A(t)x$ if $A(t)$ is peacewise constant.



$$\Pi(t,t_0) = \Pi_3(t,t_2)\,\Pi_2(t_2,t_1)\,\Pi_1(t_1,t_0)$$

$$\Pi_i(t,t_0) = e^{A_i(t-t_0)}$$

<u>Problem 3.27</u> Calculate $\Pi(t,t_0)$ for the system $\dot{x} = \begin{bmatrix} t & 0 \\ 1 & t \end{bmatrix} x = A(t)x$

$$\begin{cases} \dot{x}_1 = t x_1 \\ \dot{x}_2 = x_1 + t x_2 \end{cases}$$

$\dot{x} + p(t)x = q(t)$

$p(t) = -t$; $\quad P(t,t_0) = \int_{t_0}^{t} (-s)\,ds = -\left(\frac{t^2 - t_0^2}{2}\right)$ $\quad x_1(t) = e^{\left(\frac{t^2-t_0^2}{2}\right)} x_{01}$

$\dot{x}_2 - t x_2 = x_1$

$$x_2(t) = e^{\frac{t^2-t_0^2}{2}} x_{02} + e^{\frac{t^2-t_0^2}{2}} \int_{t_0}^{t} \cancel{e^{-\left(\frac{s^2-t_0^2}{2}\right)}} x_{01} \cancel{e^{\left(\frac{s^2-t_0^2}{2}\right)}}\,ds$$

$$= e^{\frac{t^2-t_0^2}{2}} x_{02} + (t-t_0) e^{\frac{t^2-t_0^2}{2}}$$

$$\Pi(t,t_0) = \begin{bmatrix} e^{\frac{t^2-t_0^2}{2}} & 0 \\ (t-t_0)e^{\frac{t^2-t_0^2}{2}} & e^{\frac{t^2-t_0^2}{2}} + (t-t_0)e^{\frac{t^2-t_0^2}{2}} \end{bmatrix}$$

Problem 3.31 $\dot{x} = A(t)\,x$

$\alpha_0(t)$ is the largest eigenvalue of $\frac{1}{2}(A(t) - A^T(t))$

Show that $\|\Pi(t,t_0)\| \leq e^{\int_{t_0}^{t} \alpha_0(s)\,ds}$

Hint: compute and estimate $\frac{d}{dt}(x(t))^2$ for $x(t) = \Pi(t,t_0)\,x_0$

Problem 3.39 Calculate monodromy matrix and Floquet exponents for the 2-dim system $\dot{x} = a(t)A\cdot x$, where $a(t)$ has a period T; A constant matrix.

$$\frac{dx}{dt} = a(t)\,A\,x$$

$$\frac{dx}{A\,dt} = A(x);$$ find $d\tau = a(t)\,dt$ - find a new "time" variable

$$\tau = \int_{t_0}^{t} a(t)\,dt; \Rightarrow \frac{dx}{d\tau} = Ax, \quad A \text{ - constant matrix}$$

$$x(\tau) = e^{\tau A} x_0$$

$$x(\tau = 0) = x_0 = x(t = t_0)$$

$$x(t) = e^{\int_{t_0}^{t} a(s)\,ds\, A} x_0$$

$$M(t_0) \overset{def.}{=} \Pi(t_0 + T, t_0)$$

$$M(t_0) = e^{A\int_{t_0}^{t_0+T} a(s)\,ds} \qquad \int_{t_0}^{t_0+T} a(s)\,ds = T\cdot\bar{a}$$

$\bar{a}$ ← average value of a on one period

$$M(t_0) = e^{\bar{a}TA}$$

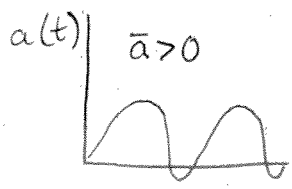
$$\Pi(t,t_0) = P(t,t_0)\,e^{Q(t_0)(t-t_0)} \quad \text{(Floquet)} \qquad P(t,t_0) \begin{cases} \text{- periodic and} \\ \text{- bounded} \end{cases}$$

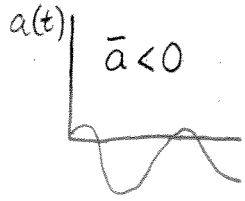
$$P(t,t_0) = e^{A\int_{t_0}^{t}(a(s) - \bar{a})\,ds} \qquad Q(t_0) = A\cdot\bar{a};\ M(t_0) = e^{TQ(t_0)}$$

<u>Floquet exponents</u> are eigenvalues of the matrix $Q(t_0)$.

If A has eigenvalues λ_1, λ_2 then $Q(t_0)$ has eigenvalues $\bar{a}\lambda_1;\ \bar{a}\lambda_2$. $\gamma_1 = \bar{a}\lambda_1$

$\gamma_2 = \bar{a}\lambda_2$





Problem 3.40. Compute monodromy matrix for the system with the following periodic matrix $A(t)$ with period 1.

$$A(t) = \begin{cases} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} = A_1 , & 0 \le t < \frac{1}{2} \\ \begin{bmatrix} \alpha & 0 \\ 1 & \alpha \end{bmatrix} = A_2 , & \frac{1}{2} \le t \le 1 \end{cases}$$

$$e^{A_1 t} = e^{\alpha t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad ; \quad e^{A_2 t} = e^{\alpha t} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

(formula 3.42 p. 70 or follows from 3.19 p. 63)

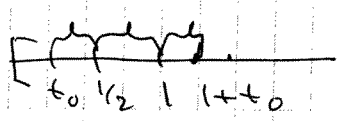
Principal matrix solution can be combined of these to expressions.

$$\Pi(t, t_0) = e^{A_1 (t - t_0)} \quad \text{if } t, t_0 \in [0, 1/2)$$

$$\Pi(t, t_0) = e^{A_2 (t - t_0)} \quad \text{if } t, t_0 \in [\tfrac{1}{2}, 1]$$

Then for $M(t_0) = \Pi(t_0 + 1, t_0)$

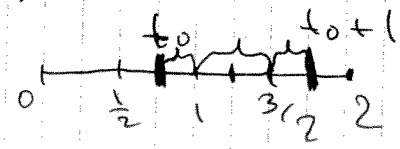
For $t_0 \in [0, 1/2]$



$$M(t_0) = \exp(A_1(t_0 + 1 - 1)) \exp\left(A_2\left(1 - \tfrac{1}{2}\right)\right) \cdot \exp\left(A_1\left(\tfrac{1}{2} - t_0\right)\right)$$

$$= \exp(A_1 t_0) \exp\left(A_2 \cdot \tfrac{1}{2}\right) \exp\left(A_1\left(\tfrac{1}{2} - t_0\right)\right)$$

For $t_0 \in [1/2, 1]$



$$M(t_0) = \exp\left(A_2\left(t_0 + 1 - 3/2\right)\right) \exp\left(A_1\left(\tfrac{3}{2} - 1\right)\right) \times$$

$$\times \exp\left(A_2(1 - t_0)\right) =$$

$$= \exp\left(A_2\left(t_0 - \tfrac{1}{2}\right)\right) \exp\left(A_1 \cdot \tfrac{1}{2}\right) \exp\left(A_2(1 - t_0)\right)$$

The interval between t_0 and $t_0 + 1$ is divided in three intervals with different expressions for $\Pi(t, t_0)$. They are combined to give $M(t_0) = \Pi(1 + t_0, t_0)$