p. 226. Periodic solutions. (1

Example. Consider the system.

$$\begin{cases} \dot{x} = -y + f(r) \cdot x \\ \dot{y} = x + f(r) \cdot y \end{cases}$$

$r = \sqrt{x^2 + y^2}$; In polar coordinates.
$x = r\cos\theta$; $y = r\sin\theta$, the system reads:

$\dot{r} = r f(r)$; $\dot{\theta} = 1$.

We observe it by differentiating

$$(r^2)' = 2r \cdot r' = (x^2+y^2)' =$$

$$= 2x\dot{x} + 2y\dot{y} = 2x(-y + f(r)x) + $$
$$+ 2y(x + f(r) \cdot y) =$$
$$= f(r)(2x^2 + 2y^2) = f(r) r^2 \cdot 2$$

The equation for θ we get by differentiating $(\tan(\theta))' = \frac{1}{\cos^2\theta} \cdot \dot{\theta} =$

$$= \left(\frac{y}{x}\right)' = \frac{y' \cdot x - x' \cdot y}{x^2} = \frac{x^2 \cancel{+ f(r)xy}}{x^2} -$$

$$- \frac{-y^2 \cancel{+ f(r)xy}}{x^2} = \frac{x^2+y^2}{x^2} = \frac{1}{(\cos\theta)^2}$$

$\Rightarrow \dot{\theta} = 1$;

Each positive zero r_0 of f corresponds to a periodic orbit. This orbit will attract trajectories nearby if $f'(r_0) < 0$ for $t \to +\infty$. (and for $t \to -\infty$ if $f'(r_0) > 0$).

If r_0 is a root with the first term in Taylor series $a(r-r_0)^2$, $a>0$ then trajectories will be attracted to the periodic orbit from inside and leave it from outside.

Polar coordinates often help, but not always.

Another useful instrument to identify periodic solutions is Poincaré map. One chooses a curve (in the plane for 2-dimensional systems), or a surface in the $\mathbb{R}^3$ such that at each point of the curve (surface) trajectories of the system cross it. It means that the vector field $f(x)$ corresponding to the system $\dot{x}=f(x)$ has everywhere $n(x)\cdot f(x)\neq 0$ on the curve (surface) with normal $n(x)$ in the point x (not tangential) - f is transversal to the curve (surface).

In higher dimensions ($\mathbb{R}^n$) one uses notion of a submanifold of codimension one.

Definition Poincaré map for a curve (surface) Γ is a mapping $P:\Gamma\to\Gamma$ such that $P(x)=\Phi_x(t_{next})$

for $x \in \Gamma$ and t_{next} is the first time when the trajectory $\Phi_x(t)$ starting at x crosses Γ in the same direction again.

It is clear that fixed points $x = P(x)$ of the Poincare map lay on periodic orbits of the system.
It makes it interesting to consider mappings of manifold into themself and their fixed points.

A simpler criterium for existence of periodic orbits gives
Theorem by Poincaré and Bendixson.
Lemma 7.13 in the book. Consider $\dot{x} = f(x)$ in $\mathbb{R}^2$
If $\omega_\sigma(x) \neq \emptyset$ $(\sigma = \pm)$ is compact and contains no fixed point, then $\omega_\sigma(x)$ is a regular periodic orbit.
In particular if C ($C \subset \mathbb{R}^2$) is a compact positively invariant set without fixed points ($f(x_0) \neq 0$) then C must contain at least one periodic orbit.

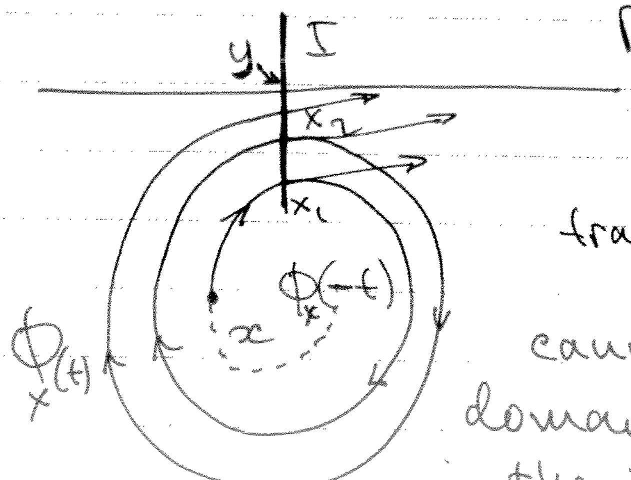
Corollary (Lemma 7.17) Let Γ be a periodic orbit of a system $\dot{x} = f(x)$ and the domain M of f includes the whole region U enclosed by Γ.

<u>Then U must contain a fixed point.</u>

<u>Proof</u> Suppose first that Γ does not include any fixed points and no periodic orbits. The closure of U is an invariant set both in positive and in negative time direction. Trajectories cannot cross the orbit Γ, because of uniqueness and Γ divides plane $\mathbb{R}^2$ into interior and exterior. (Jordans lemma).

Therefore Γ is both $\omega_+(x)$ and $\omega_-(x)$ limit set for any $x \in U$

Consider a curve I crossing Γ in point y from inside and consider a trajectory $\Phi_x(t)$, $x \in U$ and consequtive points where $\Phi_x(t)$ crosses I: $x_1, x_2, x_3 \ldots \to y \in \Gamma$



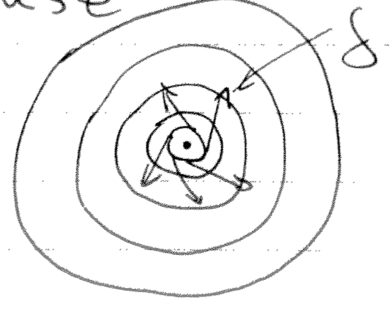
It is easy to observe that the "negative" trajectory $\Phi_x(-t)$ cannot leave the domain bounded by the interval between points x_1 and x_2 and by the part of $\Phi_x(t)$

between x_1 and x_2. Therefore Γ cannot be $\omega_+(x)$ and $\omega_-(x)$ set at the same time, because the "negative" trajectory $\Phi_x(-t)$ cannot "reach Γ.

Our hypothesis was wrong.

Therefore Γ must enclose a fixed point or another periodic orbit γ. In the second case we can continue the same argument again and get a fixed point or alternatively an infinite sequence of periodic orbits $\{\gamma_n\}$ inside each other and converging to a point x_0.

Directions of velocities on each periodic orbit γ_n must attain all values on a unit circle. Therefore the only possibility is that $\delta(x) \xrightarrow[x \to x_0]{} 0$ and $\delta(x_0) = 0$ because δ is continuous.



Problem 7.11 Bendixson's criterion.
Suppose $\operatorname{div}(f)$ does not change sign in a simply connected (open) region $U \subset M$ for a system $\dot{x} = f(x)$; M - open, $M \subset R^2$, $f \in C^1(M, R^2)$.
Then there are no regular periodic orbits contained entirely in U.

Proof. Remind Greens theorem.

$$\oint_{\partial\Omega} P\,dx_1 + Q\,dx_2 = \iint_{\Omega} \frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\, dx_1 dx_2$$

(integration in positive direction along $\partial\Omega$ - boundary of the region Ω).

Suppose that there is a periodic trajectory $(x_1(t), x_2(t))$ with period T in U.

Let $P = -f_2$, $Q = f_1$ and use Greens formula. $x_1(t)$ and $x_2(t)$ are C^1 functions.

$$\oint_{\partial\Omega} f_1\,dx_2 - f_2\,dx_1 = \int_0^T (f_1 \underbrace{\dot{x}_2}_{=f_2} - f_2 \underbrace{\dot{x}_1}_{=f_1})\,dt = 0$$

$$= \iint_{\Omega} \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\, dx_1 dx_2 = \iint_{\Omega} \operatorname{div} f\, dx_1 dx_2$$

där $\partial\Omega = \gamma(x)$ - orbit of periodic trajectory, Ω - interior of $\gamma(x)$. We wrote Greens formula here for the case when the trajectory goes counterclockwise. Otherwise it will be $-$ in front of $\operatorname{div} f$.

Function under the double integral does not change sign. Therefore this integral cannot be zero. It is contradiction with our hipothesis. ▨ Problem 7.12 - is solved similarly.

Problem 7.10. Suppose $\operatorname{div} f = 0$ in a simply connected domain.
Show that there is a function $F(x)$ such that $f_1(x) = \frac{\partial F}{\partial x_2}$; $f_2(x) = -\frac{\partial F}{\partial x_1}$ and for any orbit $\gamma(x)$, $F(\gamma(x)) = \text{const}$.

$$\oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \iint_{\Omega} \operatorname{div} f = 0$$

if Ω - simply connected, therefore the integral

$$\oint_a^b f_1 dx_2 - f_2 dx_1$$ does not depend on the path between a and b.

Therefor there is a potential F

$$F(x_1, x_2) = \int_a^{(x_1,x_2)} f_1 dx_2 - f_2 dx_1$$ such that $\frac{\partial F}{\partial x_1} = -f_2$; $\frac{\partial F}{\partial x_2} = f_1$

For any trajectory $\phi(t)$

$$F(\phi(t)) = \int_a^{\phi(t)} f_1 dx_2 - f_2 dx_1$$ and

$$\left(F(\phi(t))\right)' = \nabla F \cdot \dot{\phi} = \nabla F \cdot f = -f_1 f_2 + f_2 f_1 = 0.$$

Therefore F is constant on trajectories and orbits.

Problem Show that the equation $\ddot{x} - \dot{x}(1-3x^2-2\dot{x}^2)+x=0$ has a periodic solution.

Rewrite the problem as a system of two equations.

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -x_1 + x_2(1-3x_1^2-2x_2^2)$$

In polar coordinates

$$\dot{r} = r\sin^2\theta(1-3r^2\cos^2\theta - 2r^2\sin^2\theta)$$

$$\dot{\theta} = -1 + \frac{1}{2}\sin(2\theta)(1-3r^2\cos^2\theta-2r^2\sin^2\theta)$$

Observe that

a) for $r=\frac{1}{2}$

$$\dot{r} = \frac{1}{4}\sin^2\theta\left(1-\frac{1}{2}\cos^2\theta\right) \geq 0$$

(zero only $\theta=0$, $\theta=\pi$) $\Rightarrow$

The set $\{x : r > \frac{1}{2}\}$ is positively invariant.

b) $\dot{r} \leq r\sin^2\theta(1-2r^2)$

$\Rightarrow$ for $r < 1/\sqrt{2}$ $\dot{r} \leq 0$, $=$ only for $\theta=0$, $\theta=\pi$. $\Rightarrow \{x \mid r < 1/\sqrt{2}\}$ is positively invariant.

$\Rightarrow$ The ring $\{x : \frac{1}{2} < r < 1/\sqrt{2}\}$ is positively invariant. The only fixed point is in the origin. $\Rightarrow$

There is a periodic orbit in the ring. (by Poincare-Bendixson).