Home assignment 1 for the course ODE and mathematical modelling MVE162 in year 2018.
Two models of biological competition.

The modeling assignment consists of a theoretical part that requires some mathematical reasoning and an implementation part including writing a simple Matlab code solving an ODE, graphical output, analysis of numerical solutions and conclusions.

You are encouraged to work in small groups of 2-3 people, but each of you must write an own report. The report is supposed include ***1) theoretical argumentation, with necessary references 2) numerical results with graphical output and 3) interpretation of results, and names of all group members. Second year batchelor students attend a lecture on scientific writing by Elin Götmark on the 12-th of April and will have possibility to discuss the scientific writing aspects of the preliminary version of reports in Swedish at a workshop under supervision of Elin Götmark in the 4-th week of the course.***

Send final reports and Matlab codes with clear comments by e-mail to Alexei Heintz, heintz(at)chalmers.se. Batchelor students send reports also to Elin Götmark, *elin(at)chalmers.se. Deadline for delivering reports is Monday 30/4.*

Grades for your reports on two home assignments will contribute 16% each to the final marks for the course.

Logistic equation and two species competition model.

Let $x_i(t), i = 1, 2.$ be populations of two species. Each of the species grows with intrinsic growth rate r_i in case when infinite resources are available: $x_i' = r_i x_i$, $r_i > 0$.

Limited resources lead to competition within the population and a limited growth rate for large size of the population: $r_i(1 - \frac{x_i}{K_i})$. This model is called the logistic equation:

$$x_i' = r_i x_i \left(1 - \frac{x_i}{K_i}\right) \tag{1}$$

The competition between different species leads a decrease in each population with the decreasing rate proportional to the competitor population size: $-\alpha_1 x_2$ for the population x_1 and $-\alpha_2 x_1$ for the population x_2 with competition coefficients $\alpha_1 > 0$ and $\alpha_2 > 0$. The corresponding system of equations describes the evolution of two competing species:

$$\begin{aligned} x_1' &= r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 \\ x_2' &= r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - \alpha_2 x_2 x_1 \end{aligned} \tag{2}$$

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Working plan and questions for the assignment N1

1. Discribe the phase portrait of the equation (1) without solving the equation analytically (which is possible).

2. Illustrate your theoretical conclusions about solutions of (1) by a graph with several representative curves for different initial data and give a biological interpretation of your mathematical conclusions.

2. Show that solutions I.V.P. for (2) starting from positive initial data for number of species $x_1(0) > 0$, $x_2(0) > 0$, will always have positive components for $t > 0$.

3. Prove that all solutions to (2) corresponding to $x_1(0) > 0$, $x_2(0) > 0$ always stay bounded.

4. Show that the maximal existence interval for solutions to the system (2) starting at $t = 0$ with $x_1(0) > 0$, $x_2(0) > 0$, is $\mathbb{R}_+$ (a useful argument was introduced in a similar situation on a lecture in the second week of the course).

5. Consider all qualitatively different positions of isoclines (lines where $x_1' = 0$ or $x_2' = 0$) of the system (2) for various relations between the parameters $\frac{r_1}{\alpha_1}$, K_1 and $\frac{r_2}{\alpha_2}$, K_2 (there are four distinct cases). Find corresponding equilibrium points. Draw pictures with isoclines illustrating each case.

6. Investigate stability properties of all equilibrium points in (2) depending on parameters for each of the qualitatively different combinations of parameters.

7. Illustrate your theoretical conclusions about different scenarios of the evolution for (2) by pictures with phase portraits and isoclines for each of the cases.

8. Give a biological interpretation to each of the four possible scenarios of evolution by (2).

9. Is there a qualitative difference between the behaviour of solutions to the logistic equation and to the two species competition model when $t \to \infty$? How this difference depends on if α_1 and α_2 are small or large (with other parameters fixed)?

Hints to Matlab implementation

1. Write two Matlab codes: one that can solve a scalar ODE and one that can solve a system of two ODEs. Let both programs to choose several initial data points and draw corresponding integral curves both as functions of time. In the case of the system of two equations the code must draw trajectories of $(x_1(t), x_2(t))$ in the phase plane of (x_1, x_2) in a separate figure. You can use function *plot* for graphics. In the case of the system of two equations the function *ginput* in Matlab can be used (not obligatory) to choose a point of initial data from the plane of (x_1, x_2). Define coordinates limits in the figure at least twise larger than K_i before using *ginput* to make the set of possible initial data large enough for choosing them by *ginput*. You can learn how to use these hints from examples of Matlab codes on the home page.

2. Write a Matlab function describing the right hand side of the logistic equation.

3. Write a Matlab function describing the right hand side of the two species competition model. Remember that initial data must be a 2-dimensional column vector and the solution will be a $n \times 2$ matrix where n is a number of time points in the output.