1 Bendixson's citerium for nonexistence of periodic solutions in plane.

Theorem. Let $x' = f(x)$ with $f : G \to \mathbb{R}^2$, $G \subset \mathbb{R}^2$ be open and let $D \subset G$ be a simply connected domain (domain without "holes" even point holes). f is locally Lipschitz in G.

Suppose that $\text{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is strictly positive (or strictly negative) in D, where $f = [f_1, f_2]^T$.

Then the equation has no periodic solutions with orbits inside D.

Proof 1. Carry out a proof by contradiction. Suppose that there is a periodic trajectory $x(t)$ with period T in D. $x(t+T) = x(t)$ and

$$x_1'(t) = f_1(x(t)), \qquad x_2'(t) = f_2(x(t))$$

Denote the orbit of $x(t)$ by $\mathcal{L} = \{x(t) : \quad t \in [0,T]\}$. Denote the domain inside $\mathcal{L}$ by Ω. Then the boundary $\partial\Omega = \mathcal{L}$ because $D \supset \Omega$ is simply connected and has no holes. Consider the integral of $\text{div}(f)$ over Ω and apply Gauss theorem:

$$I = \int_{\Omega} \text{div}(f) dx_1 dx_2 = \int_{\partial\Omega} f \cdot n \, dl$$

where n is the outward normal to the boundary $\partial\Omega$. Point out that $f(x(t)) = x'(t)$ on $\partial\Omega = \mathcal{L}$ because $\mathcal{L}$ is the orbit of the periodic solution $x(t)$ that we supposed to be existing. Therefore $f(x(t))$ is the tangent vector to $\partial\Omega$ and therefore scalar product of it woth the normal vector is zero $f \cdot n = 0$. Therefore

$$I = \iint_{\Omega} \text{div}(f) dx_1 dx_2 = \int_{\partial\Omega} f \cdot n \, dl = 0$$

with the curve integral over $\partial\Omega = \mathcal{L}$ in the right hand side. On the other hand $\text{div}(f) > 0$ (or strictly negative) in the whole $D \supset \Omega$.Therefore the integral $I = \int_{\Omega} \text{div}(f) dx_1 dx_2$ over a bounded domain Ω must be strictly positive (negative).We arrived to contradiction: $0 > 0$. Therefore our supposition was wrong and the system cannot have a periodic orbit in D.■

Proof 2. starts similarly with the supposition that there is a periodic trajectory $x(t)$ with period T in D, $x(t+T) = x(t)$ and

$$x_1'(t) = f_1(x(t)), \qquad x_2'(t) = f_2(x(t))$$

Denote the orbit of $x(t)$: by $\mathcal{L}$ =$\{x(t): \quad t \in [0,T]\}$. Denote the domain inside $\mathcal{L}$ by Ω. Then the boundary $\partial\Omega = \mathcal{L}$ because $D \supset \Omega$ is simply connected and has no holes.

We apply the Greens formula:

$$\oint_{\partial\Omega} P dx_1 + Q dx_2 = \iint_{\Omega} \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dx_1 dx_2$$

instead of Gauss theorem.

Choose $P = -f_2$, $Q = f_1$ and express the contour integral in the left side of the Greens formula using the definition of the contour integral:

$$\oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \int_0^T (f_1 x_2' - f_2 x_1') \, dt$$

Point out that $x_1'(t) = f_1(x(t))$ and $x_2'(t) = f_2(x(t))$ and substitute these expressions into the integral:

$$\oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \int_0^T (f_1 f_2 - f_2 f_1) \, dt = 0$$

Apply the Greens formula substitute expressions for P and Q, and conclude that in the case $\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) = \operatorname{div}(f) > 0$:

$$0 = \oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \iint_{\Omega} \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 > 0$$

that is contradiction: $0 > 0$. In the case if $\operatorname{div}(f) < 0$ in D we arrive to the contradiction < 0. ■

1. **Example.**

Show that the following system of ODE has no periodic solutions.

1. $$\begin{cases} x' = x^3 - y^2 x + x \\ y' = -0.5y + y^3 + x^4 y \end{cases}$$

   We consider divergence of the right hand side of the system.

   $\text{div}(f) = 3x^2 - y^2 + 1 - 0.5 + 3y^2 + x^4 = x^4 + 3x^2 + 2y^2 + 0.5 > 0$

Therefore divergence of the right hand side of the equation is positive everywhere in the plane that is a simply connected set (does not have holes, even point-holes). According to Bendixson's criterion the system cannot have periodic solutions anywhere in the plane.

Example.

Show that the following system of ODE has no periodic solutions.

1. $$\begin{cases} x' = \frac{1}{7} + x^2 - yx + y^2 \\ y' = -\frac{1}{5} - y^2 \end{cases}$$

   Solution

   y' is always strictly negative. It implies that $y(t)$ must be monotone function of time. It contradicts to possibility of having periodic solutions that are always bounded.