List of notions, methods, theorems and typical problems to examination in ODE and Mathematical Modeling MMG511/TMV162, year 2018.

(will be slightly updated with examples and references during the course)

References are given to pages in the course book by Logemann and Ryan.

One must know:

- **definitions to all notions,**
- **all formulations of the theorems from the list,**
- **must be able to prove theorems marked by green (or grey in black-white version),**
- **must be able to solve problems of the types mentioned in the list and to make conclusions from the theory.**

Topics, definitions and notions	Methods, theorems, lemmas and corollaries	Typical problems
Preliminary notions form linear algebra and analysis Vector space, normed vector space, norm of a matrix. Cauchy sequence. Complete vector space (Banach space) Compact sets i R^n. Continuous functions. Uniform convergence in the space of continuous functions.	**Background results from analysis** • Space C(I) of continuous functions on a compact I is a Banach space Example A.14, p. 272 • Bolzano-Weierstrass theorem. Theorem A.16, p. 273 Weierstrass criterion on uniform convergence of functional series. Corollary A23 ,p. 277	
Introduction 1.2 Initial value problem (I.V.P.), p.13 existence, uniqueness, Maximal solution. p. 13 Integral form of I.V.P. pp.16-17 Classification of ODEs: order, autonomous, non - autonomous, linear, non-linear	Elementary Examples 1.1-1.2, pp. 13-14 on existence, uniqueness, maximal existence time for solutions showing a blow up, global solutions p.15 Elementary solution methods for 1-dimensional ODEs of first order: linear ODEs pp.18-19, Ex. 1.5 ODEs with separable variables p. 15, Ex. 1.3	Solve an ODE with separate variables or a linear ODE of first order. Find maximal existence time interval for an explicit solution.

LINEAR SYSTEMS

Topics, definitions and notions	Methods, theorems, lemmas and corollaries	Typical problems
Preliminary notions form linear algebra Vector space, normed vector space, norm of a matrix. **General linear systems §2.1** x'=A(t)x **Transition matrix function $\Phi(t,\tau)$ §2.1.1** Solution space, p.30. Fundamental system and fundamental matrix solution. Def. p.32 Wronskian	**A background result from analysis** Weierstrass criterion on uniform convergence of functional series. Corollary A23 ,p. 277 **Transition matrix function and fundamental matrix solutions and their properties.** • The construction of transition matrix function $\Phi(t, \tau)$. Lemma 2.1, p.24 • **Gronwall's inequality,** Lemma 2.4, p. 27 • A simple version of Gronwall's inequality, and uniqueness of solutions to I.V.P. of linear ODE. Th. 2.5, p.28 (we used only a simple version of Grönwall inequality on lectures, with constant under the integral by taking max of $\lVert A(s)\rVert$ under the integral, and repeated the same argument as for autonomous ODE that was studied earlier. Corollary 2.9, p.34, • Group properties of transition matrix function. Corollary 2.6, p.29 • On the **dimension of the space of solutions to a linear system of ODEs.** Prop. 2.7 statements (1) and (3), p.30. • Abel's formula: Formula 2.14 in Proposition 2.7, p. 30, • Fundamental matrix for linear homogeneous ODE and its connection with transition matrix function Prop. 2.8, p. 33 • **Variation of constant (Duhamel's) formula** $x(t) = \Phi(t,\tau)\xi + \int_\tau^t \Phi(t,\sigma)b(\sigma)d\sigma$ Form. 2.27, Th. 2.15, p.41- for general non-homogeneous linear ODE.	Find principle matrix solution for a simple system of ODE that can be solved explicitly (for example with triangular matrix)
§2.1.3 Linear system of ODE with constant matrix (autonomous systems) x'=Ax Matrix norm, formula A.10, A.11, p. 278	**Preliminary properties of block matrices and similar matrices.** • Polynomial of block diagonal matrices.	**Typical problems for linear autonomous systems** Find general solution or solve I.V.P. for a linear autonomous system of ODE with constant matrix in case when eigenvalues are given or are easy to

Linear change of variables in ODE. Matrices $B = P^{-1}AP$ and A are called similar Polynom P(A), exponent exp(A) and logarithm log(A) of a matrix A. Arbitrary functions of matrices Diagonalizable matrices, Block diagonal matrices. Algebraic and geometric multiplicity of eigenvalues pp. 268-269 **Generalized eigenspaces and generalized eigenvectors** p. 267 Chains of eigenvectors (see lecture notes) **Jordan canonical form of matrix** J , p. 268 Jordan block: p. 268 Jordan canonical form of a matrix. Transformation leading to the Jordan canonical form J of matrices: T^(-1)AT=J, A=TJT^(-1).	• Determinant and eigenvalues of block triangular matrices. • For two similar matrices A and $J = T^{-1}AT$ determinant, characteristic polynomial, eigenvalues, and trace Tr(A) are the same. • Property of matrix norm:||AB||<=||A||||B||, A.12, p. 279 • Properties of exp(A), Lemma 2.10, without (2) p.34, in particular: for two commuting matrices: AB=BA it follows that exp(A+B)=exp(A)exp(B) • Functions of two similar matrices A ans B are expressed explicitly by each other for example: **for B=TAT^(-1); exp(B)=Texp(A)T^(-1),** see p. 62 - in the proofs to Th, 2.19 and 2.29, pp. 60 and 62, • The solution for linear systems of ODE with constant matrix and initial condition $x(\tau) = \xi$ is: $x(t) = \exp((t-\tau)A\xi$. • A simple version of Gronwall's inequality with constant coefficient under the integral, and uniqueness of solutions to I.V.P. for linear autonomous ODE. Corollary 2.9, p. 34. We use a similar proof for the Th. 2.5, p. 28. in lecture notes • Theorem A.8, p.268 on generalized eigenspaces and basis of eigenvectors and generalized eigenvectors. • Method to find a basis of generalized eigenvectors. • Theorem A.9, p.268 on Jordan canonical form of a matrix. • **Connection J=T^(-1)AT between a matrix A and its Jordan canonical form J in terms of eigenvectors and generalized eigenvectors to A**. See lecture notes. • Number of blocks in the Jordan form of matrix is equal to the number of linearly independent eigenvectors. • **Structure of the general solution to linear ODE with constant coefficients:** Theorem 2.11, p.35 • Function of a Jordan block: formula. (2.47), p.61, - two important particular cases are: the f(J)=exp(J) and f(J)=log(J); exponential function and for logarithm – see lecture notes. • Theorem 2.12 and **Corollary 2.13, p. 36 on stability and asymptotic stability of solutions to linear autonomous systems** of ODEs. The proof to Corollary 2.13 is the same as one to the Theorem 2.12, but a bit simpler because of the more transparent formulation of the Corollary. Alternative proof is available in lecture notes. • Classification of phase portraits in plane for linear systems with constant matrix, see the link on the homepage and lecture notes. • **Variation of constant (Duhamel's) formula** in Corollary 2.17, p.43 for non - homogeneous linear systems with constant matrix	calculate (use Theorem 2.11, p.35 and hints in the exercises on the homepage) Solve a non - homogeneous linear system of ODEs $x'(t) = Ax + b(t)$ using **Duhamel's formula in Corollary 2.17** p.43 $x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-\sigma)}\, b(\sigma)d\sigma$ Decide if a vector valued function can be solution to a linear system of ODEs just by checking it´s structure. Find a basis of generalized eigenvectors to a matrix. Use Theorem 2.11 to find if all solutions to a particular linear autonomous systems that are bounded or tend to zero with t going to plus infinity. Use general solution to a linear autonomous system to find for which initial data solutions are bounded or tend to zero with t going to plus infinity. Compute exponent of a 2x2 matrix or a block diagonal matrix with eigenvalues that are easy to guess. Compute exponent of an arbitrary Jordan matrix. Consider a 2- dimensional linear system in plane: classify and draw a sketch of phase portrait.
Linear systems with periodic coefficients. Logarithm and principal value of logarithm for complex numbers. log(z)=log(|z|)+iArg(z) (meaning natural logarithm here) Logarithm and principal logarithm of a matrix. p. 52 **Monodromy matrix** - the notion is not used in the book, but is introduced without this name as Φ(p,0) value of the transition matrix Φ(t, τ) in Floquet theory for linear systems with periodic matrix A(t+p)=A(t). **Floquet (characteristic) multipliers** are eigenvalues to the monodromy matrix Φ(p,0): Definition p. 48	• Fundamental matrix of periodic linear system with period p is p- shift invariant: Formulas 2.31, 2.32, p.45 • Proposition 2.20 on existence of periodic solutions to a periodic linear system • Connection between the logarithm of a matrix and the logarithm of its Jordan canonical form. • Formula for logarithm of a Jordan block. • Existence of principal logarithm of a non-degenerate matrix - Proposition 2.29. p.53. • **Floquet representation of transfer matrix for periodic systems** in Theorem 2.30, p.53. • Floquet Theorem 2.31 on the connection between the absolute values of Floquet multipliers and the boundedness and the zero limit of solutions to periodic linear systems. p. 54 • Corollary 2.33, p 59 on a criterion for existence of unbounded solutions to a periodic linear system.	Find a monodromy matrix for a simple equation that can be solved explicitly. Find if a periodic linear system has periodic solutions. Calculate Floquet multipliers for systems with separable variables where the transition matrix and the monodromy matrix can be calculated explicitely. Decide if solutions to such a system all tend to zero or stay bounded. Find using Corollary 2.33 if a periodic linear system has unbounded solutions.

	• Spectral mapping Theorem 2.19, p. 44, essentially in the case f(x)=exp(x) giving the connection between characteristic multipliers and eigenvalues to the logarithm of the monodromy matrix.	

NONLINEAR SYSTEMS

Topics, notions, definitions	Methods, theorems, lemmas and corollaries	Typical problems
Background notions from analysis Metric and normed vector spaces, pp.269-270 Cauchy sequence. P. 270 complete space, Banach space: p.270 Open, closed, compact, connected sets p.270 Bounded, compact, precompact sets, p. 270 Space C(I) of continuous functions on a compact I. Uniform convergence p.273 **Fixed point theorems** Fixed point of an operator. **Contraction map**. p.278 in Th. A.25 Sequence of iterations, p. 278 in the proof of Theorem A.25	**Background results from analysis** • Space C(I) of continuous functions on a compact I is a Banach space. Example A.14, p. 272 • Bolzano-Weierstrass theorem Theorem A.16, p. 273 • **Banach's contraction mapping principle**. Theorem A.25, p.277	**Exercises on contraction principle.** Show that an operator is a contraction in C(I). Show using the Banach contraction principle that a given operator has a fixed point in some ball.
Local existence and uniqueness theory for Initial Value Problem (IVP) Integral form of IVP, p. 102, p. 119 extension of solution, p. 106 maximal solution Lipschitz functions: formula 4.7, p. 115 Picard iterations, p.23	**Local existence and uniqueness theory** **Gronwall's inequality** Lemma 2.4, p. 27 Lipschitz condition and uniqueness of solutions Th. 4.17, p. 118, Th. 4.18, p. 119. Contraction mapping principle for existence and uniqueness theorem (**Picard-Lindelöf theorem**) Theorem. Th. 4.22, p.122, Steps 1 and 2 of the proof. Picard iterations (p. 23)	Identify Lipschitz functions of several variables. Use Gronwall inequality to estimate difference between solutions to an ODE with different initial data on a finite time interval. Write explicitly 2-3 Picard iterations (p. 23) for an equation. Find conditions for convergence of Picard iterations for a particular equation.
Extension (continuation) of solutions and maximal interval of existence. §4.2 Continuation (extensibility) of solution Maximal existence interval , p. 106 maximal solution, p. 106 Global solution	Nonlinear systems of ODE**, Maximal solution.** Existence of maximal solutions. Th. 4.8, p.108 Extensibility to a boundary point of the existence interval. Lemma 4.9, p. 110; Cor. 4.10, p. 111. On the size of the maximal interval Th. 4.11, p. 112 on possible limits and maximal existence intervals for maximal solutions Th. 4.25, p. 125 on possible limits and maximal existence intervals for maximal solutions Prop. 4.12, p.114, on "infinite" extensibility of solutions for ODE with linear bound on the right hand side.	Investigate if an ODE has global solutions. Decide for solutions, starting in a certain domain how long they can be extended and which limits they might have for time going to infinity. For example Examples 1.2 p. 14, 4.33 on the page 139 example 4.5, 4.6, 4.7 on pages 107-108
Transition map. Transition property Transition map or flow, for autonomous systems – translation property.	Transition map (dynamical system) Transition property, (Chapman Kolmogorov formula for non-linear systems) Theorem 4.26, pp. 126-127 The domain and continuity of transition map. Theorem 4.29, Lemma 4.30, p. 129	
Autonomous systems **Limit sets and invariant sets.** Positive, negative semi-orbits p. 141 to a flow (dynamical system). ω -limit point and α limit point, p. 142 ω -limit sets and α limit sets, p. 142 Positively invariant, negatively invariant sets, p. 142.	**Properties of limit sets.** **Properties of limit sets: ω -limit sets are connected invariant sets** Th. 4.38, p.143	Find an omega (positively) invariant set with desired properties for an ODE. Using test functions to identify positively - invariant sets to an ODE
Periodic solutions to autonomous systems in the plane Equilibrium (critical) points, p.145 periodic points, periodic orbits, non-periodic orbits, p. 146 Limit cycles are limit sets that are periodic orbits.	**Poincare - Bendixson theorem** 4.46, p. 151 (without proof).: "A limit set of a solution in a compact positively invariant set without fixed points is a periodic orbit" **Bendixsons criterion** for the non-existence of periodic solutions: div(f) >0 or div(f)<0 in a simply connected domain U - without holes (after lecture notes on the home page) First integrals and periodic orbits. §4.7.2 Prop. 4.54, p. 161: level sets of first integrals in the plane that are closed curves are periodic orbits.	Prove that an ODE has at least one periodic solution by Poincare Bendixson theorem. Prove that an ODE in plane does not have periodic solutions in a domain using Bendixson's negative criterion

Stability of equilibrium points of nonlinear systems. Chapter 4. Definitions of stable and asymptotically stable equilibrium points. Definition 5.1, p. 169 Def. 5.14, p.182. **Stability by linearization** Linearization of ODE. § 5.6, p.194	**Stability of equilibrium points of nonlinear systems.** **Stability** of autonomous non-linear ODEs **by linearization with Hurwitz variational matrix**. Th. 5.27, p.193 and Corollary 5.29, p. 195. (the proof given on the lecture uses the Gronwall inequality and is available at the homepage). **Grobman-Hartman theorem**: solutions to a nonlinear system and its linearization around an equilibrium point are "equivalent" if all real parts of eigenvalues to the variational matrix are non zero (lecture notes on the homepage without proof)	Show stability of a fixed point using Theorem 5.27, Corollary 5.29 about linearization with Hurwitz variational matrix Investigate stability of a fixed point using the Grobman- Hartman theorem about linearization.
Stability of fixed points by the method with Lyapunov functions. Lyapunov function, V(0)=0, V(x)>0 for x not 0 V_f <=0 strict Lyapunov function: the same but $V_f(x) < 0$ for x not 0	**Stability of equilibrium points** to autonomous ODE **by Lyapunovs functions:** Theorem 5.2, p. 170 A constructive variant of the proof is available on the home page. Students are free to choose any variant of the proof to Th. 5.2 at the exam. **Instability of fixed points to autonomous ODE by Lyapunovs method:** Th. 5.7, p. 174 A constructive variant of the proof to a slightly weaker theorem is available on the home page.	Show stability (asymptotic stability) of a fixed point of an ODE by Lyapunovs method. Show instability of a fixed point of an ODE by Lyapunovs method.
Invariance principles. Domain of attraction, Def. 5.19 p. 186 Globally attractive equilibrium Def. 5.21, p.187	Invariance principles. **LaSalle's invariance principle** Th.5.12, p.180; Proof in Exercise 5.9 on page 312 **Asymptotic stability by "weak" Lyapunov's functon using Krasovsky-La Salle theorem.** Th. 5.15, p. 183	 Apply -LaSalles invariance principle to show asymptotic stability of a fixed point using a "weak" Lyapunov function. Find a domain of attraction for an asymptotically stable equilibrium point. Typical problems in the book are: Example 5.13, p. 181, Exercises 5.7, 5.8 , p. 182