Exercises on general linear ODE

1. Show that $(A(t)B(t))' = A'(t)B(t) + A(t)B'(t)$ for $n \times n$ matrices $A(t)$ and $B(t)$ with differentiable elements.

2. Show that $\det(\exp(A)) = \exp(\mathrm{tr}A)$ for any constant matrix A.

3. If $t \longmapsto \Psi(t)$ is a fundamental matrix solution for the system $x' = A(t)x$, $x \in \mathbb{R}^n$. It means that $\Psi'(t) = A(t)\Psi(t)$.

Then the matrix valued function $\Phi(t, \tau) = \Psi(t)\Psi^{-1}(\tau)$ is called the transition matrix function: it is a fundamental matrix solution with respect to the variable t for each τ such that $\Phi(\tau, \tau) = I$. It implies that the solution $x(t)$ to I.V.P.

$$x' = A(t)x, \quad x(\tau) = \xi$$

with initial data ξ at the time τ is given by the expression:

$$x(t) = \Phi(t, \tau)\xi$$

The matrix $\Phi(t, \tau)$ satisfies Chapman-Kolmogorov identities:

$$\Phi(t, s)\Phi(s, \tau) = \Phi(t, \tau)$$

(semigroup property) and

$$\Phi^{-1}(t, s) = \Phi(s, t), \qquad \frac{\partial \Phi(t, s)}{\partial s} = -\Phi(t, s)A(s)$$

Prove these statements.

4. Calculate the transition matrix function $\Phi(t, s)$ for the system of equations

$$\begin{cases} x_1' = t\, x_1 \\ x_2' = x_1 + t\, x_2 \end{cases}$$

5. Calculate the transition matrix function $\Phi(t, s)$ for the system of equations

$$\begin{cases} x_1' = x_1 + t x_2 \\ x_2' = 2x_2 \end{cases}$$

6. Suppose that every solution of $x' = A(t)x$ is bounded for $t \geq 0$ and let $\Psi(t)$ be a fundamental matrix solution. Prove that $\Psi^{-1}(t)$ is bounded for $t \geq 0$ if and only if the function $t \to \int_0^t \mathrm{tr}A(s)ds$ is bounded from below. Hint: The inverse of a matrix is the adjugate of the matrix divided by its determinant. See: http://en.wikipedia.org/wiki/Adjugate_matrix

7. Suppose that the linear system $x' = A(t)x$ is defined on an open interval containing the origin whose right-hand end point is $w \leq \infty$ and the norm of every solution has a finite limit as $t \to w$. Show that there is a solution converging to zero as $t \to w$ if and only if $\int_0^w \mathrm{tr}A(s)ds = -\infty$. **Hint**: Use Abels formula and the fact that a matrix has a nontrivial kernel if and only if its determinant is zero.

7a. Show that if $\liminf_{t \to +\infty} \int_{t_0}^t tr(A(s))ds = +\infty$ then the equation $x' = A(t)x$ has an unbounded solution. Hint: use Abel's formula.

8. Let A be an invertible constant matrix. Show that the only invariant lines for the linear system $x' = Ax$, $x \in R^2$ are the lines $ax_1 + bx_2 = 0$ where $[-b, a]^T$ is an eigenvector to A.

9. Show that for arbitrary $n \times n$ matrix A the relation $\det(I + \varepsilon A + O(\varepsilon^2)) = 1 + \varepsilon\, \mathrm{tr}\,(A) + O(\varepsilon^2)$

10. Consider the flow $\phi(t, x)$ corresponding to the autonomous equation $y' = f(y)$, $y \in \mathbb{R}^n$ mapping the domain Ω to the domain as $\Omega_t = \{y = \phi(t, x), x \in \Omega\}$ where y is the solution to the ODE $y' = f(y)$ with initial data $y(0) = x \in \Omega$.

Show that $\frac{d}{dt}\left(Vol(\Omega_t)\right) = \int_{\Omega_t} \mathrm{div}\,(f)\, dy$. **Hint:** use the result of Ex.9.

11. Show directly that the area of a unit disk is preserved when it is transformed forward to 2 time units by the flow, corresponding to the system $x' = y$, $y' = x$. **Hint**: consider the system in new variables $x + y$ and $x - y$.

Solutions.

Solution to 3.

- $\Phi(t,s)\Phi(s,\tau) = \Psi(t)\Psi^{-1}(s)\Psi(s)\Psi^{-1}(\tau) = \Psi(t)\Psi^{-1}(\tau) = \Phi(t,\tau)$.
- $\Phi^{-1}(t,s) = \left(\Psi(t)\Psi^{-1}(s)\right)^{-1} = \left(\Psi^{-1}(s)\right)^{-1}(\Psi(t))^{-1} = \Psi(s)\Psi^{-1}(t) = \Phi(s,t)$,
- $\frac{\partial\Phi(t,s)}{\partial s} = -\Phi(t,s)A(s)$

  Use the relation: $\frac{d}{ds}\left(\Psi^{-1}(s)\right) = -\Psi^{-1}(s)\frac{d}{ds}\left(\Psi(s)\right)\Psi^{-1}(s)$

  $\frac{\partial\Phi(t,s)}{\partial s} = \frac{\partial\left(\Phi^{-1}(s,t)\right)}{\partial s} = \left(-\Phi^{-1}(s,t)\frac{\partial}{\partial s}\left(\Phi(s,t)\right)\Phi^{-1}(s,t)\right) = -\Phi^{-1}(s,t)A\,\Phi(s,t)\Phi^{-1}(s,t) = -\Phi^{-1}(s,t)A = -\Phi(t,s)A$

Solution to 4.

Solution to the scalar linear equation $x' = p(t)x + g(t)$ with initial data $x(\tau) = x_0$ is calculated using the primitive function $\mathbb{P}(t,\tau)$ of $p(t)$.

$$
\begin{aligned}
x' &= p(t)x + g(t) \\
\mathbb{P}(t,\tau) &= \int_\tau^t p(s)ds \\
x(t) &= \exp\{\mathbb{P}(t,\tau)\}\,x_0 + \int_\tau^t \exp\{\mathbb{P}(t,s)\}\,g(s)ds \\
x(\tau) &= x_0
\end{aligned}
$$

The system of ODE above has triangular matrix and can be solved recursively starting from the first equation.

The fundamental matrix $\Phi(t,s)$ has columns π_1 and π_2 that at the time τ have initial values $[1,0]^T$ and $[0,1]$, because $\Phi(\tau,\tau) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

In the equation $x_1' = t\,x_1$ the coefficient $p(t) = t$, therefore $\mathbb{P}(t,\tau) = \int_\tau^t s\,ds = \left(\frac{1}{2}s^2\right)\Big|_\tau^t = \frac{1}{2}\left(t^2 - \tau^2\right)$ and the solution $x_1(t) = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))x_1(\tau)$.

The second equation $x_2' = t\,x_2 + x_1$ is similar but inhomogeneous: $x_2(t) = \exp(\mathbb{P}(t,t_0))x_2(t_0) + \int_{t_0}^t \exp(\mathbb{P}(t,s))x_1(s)ds$.

Substituting $\mathbb{P}(t,\tau) = \frac{1}{2}\left(t^2 - \tau^2\right)$ we conclude that $x_2(t) = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))x_2(\tau) + \int_\tau^t \exp(\frac{1}{2}\left(t^2 - s^2\right))\exp(\frac{1}{2}\left(s^2 - \tau^2\right))x_1(\tau)ds = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))x_2(\tau) + \int_\tau^t \exp(\frac{1}{2}\left(t^2 - \tau^2\right))x_1(\tau)ds$

And

$x_2(t) = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))x_2(\tau) + \exp(\frac{1}{2}\left(t^2 - \tau^2\right))x_1(\tau)(t - \tau)$. The fundamental matrix solution $\Phi(t,\tau)$ has columns that are solutions to $x' = A(t)x$ with initial data - that are columns in the unit matrix: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

Taking $x_1(\tau) = 1$ and $x_2(\tau) = 0$ we get $x_1(t) = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))$ with $x_2(t) = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))(t - \tau)$

Taking $x_1(\tau) = 0$ and $x_2(\tau) = 1$ we get $x_1(t) = 0$ with $x_2(t) = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))$ and the fundamental matrix solution in the form

$$\Phi(t,\tau) = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))\begin{bmatrix} 1 & 0 \\ t - \tau & 1 \end{bmatrix}$$

Solution to 5. The solution is similar to the problem 4.

$$
\begin{aligned}
x' &= p(t)x + g(t) \qquad (1)\\
\mathbb{P}(t,t_0) &= \int_{t_0}^{t} p(s)ds\\
x(t) &= \exp\left\{\mathbb{P}(t,t_0)\right\} x_0 + \int_{t_0}^{t} \exp\left\{\mathbb{P}(t,s)\right\} g(s)ds\\
x(t_0) &= x_0
\end{aligned}
$$

$\begin{cases} x_1' = x_1 + tx_2 \\ x_2' = 2x_2 \end{cases}$. $x' = Ax$, $A = \begin{bmatrix} 1 & t \\ 0 & 2 \end{bmatrix}$

$$
\begin{aligned}
\Phi(t,\tau) &= (\pi_1(t,\tau), \pi_2(t,\tau)).\\
\frac{\partial}{\partial t}\pi_1 &= A\pi_1; \qquad \frac{\partial}{\partial t}\pi_2 = A\pi_2
\end{aligned}
$$

$\pi_1(\tau,\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\pi_2(\tau,\tau) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We solve first the equation for $x_2(t)$ with initial data $x_2(\tau)$:

$$x_2(t) = x_2(\tau)\exp(2(t-\tau))$$

and then the equation for $x_1(t)$ with initial data $x_1(\tau)$ and substituting the solution for $x_2(t) = x_2(\tau)\exp(2(t-\tau))$into the right hand side of the equation, both accoring to the formula in (1)

$$
\begin{aligned}
x_1(t) &= x_1(\tau)\exp(t-\tau) + \int_{\tau}^{t} \exp(t-s)\left[s\, x_2(\tau)\exp(2(s-\tau))\right] ds\\
&= x_1(\tau)\exp(t-\tau) + x_2(\tau)\exp(t-2\tau)\int_{\tau}^{t} \exp(s) s ds =\\
&\quad \left\{\int_{\tau}^{t} \exp(s) s ds = te^{t} - \tau e^{\tau} - (e^{t} - e^{\tau})\right\}\\
&= x_1(\tau)\exp(t-\tau) + x_2(\tau)\left(e^{t-\tau} - \tau e^{t-\tau} - e^{2(t-\tau)} + te^{2(t-\tau)}\right)
\end{aligned}
$$

and substitute particular initial data for $\pi_1(t,\tau), \pi_2(t,\tau)$:

$$
\Phi(t,\tau) = \begin{bmatrix} \exp(t-\tau) & \exp(t-\tau)(1-\tau) + \exp(2\,(t-\tau))(t-1) \\ 0 & \exp(2(t-\tau)) \end{bmatrix}
$$

Solution to 6.

Suppose that every solution of $x' = A(t)x$ is bounded for $t \geq 0$ and let $\Psi(t)$ be a fundamental matrix solution. Prove that $\Psi^{-1}(t)$ is bounded for $t \geq 0$ if and only if the function $t \to \int_0^t \mathrm{tr}A(s)ds$ is bounded from below. Hint: The inverse of a matrix is the adjugate of the matrix divided by its determinant, namely $\Psi^{-1}(t) = [\det(\Psi(t))]^{-1}\,[Adj(\Psi(t))]$. The adjugate $Adj(B) = \left(\widetilde{B}\right)^{T}$ where the matrix $\widetilde{B}$ is a matrix of the same size as B with elements in $\widetilde{B}_{ij}$ calculated as $n-1$ dimentional determinants of the matrix B with eliminated i-th row and j-th column times $(-1)^{i+j}$. See http://en.wikipedia.org/wiki/Adjugate_matrix

The fact that all solutions to the ODE are bounded for $t \geq 0$ implies that all elements in $\Psi(t)$ are bounded for $t \geq 0$ and therefore all elements of $Adj(\Psi(t))$ are bounded for $t \geq 0$ since they consist of sums of products of bounded elements in $\Psi(t)$ times ± 1. It implies that $\Psi^{-1}(t)$ is bounded (has bounded elements) for $t \geq 0$ if and only if $[\det(\Psi(t))]^{-1}$ is bounded that is equivalent to that $|\det(\Psi(t))|$ is bounded from below for for $t \geq 0$. Abel's formula gives that $\det(\Psi(t)) = \det(\Psi(0))\exp\left(\int_0^t \mathrm{tr}A(s)ds\right)$ and that $|\det(\Psi(t))| = |\det(\Psi(0))|\exp\left(\int_0^t \mathrm{tr}A(s)ds\right) > a > 0$, (bounded from below) if and only if $\ln\left(|\det(\Psi(0))|\right) + \left(\int_0^t \mathrm{tr}A(s)ds\right) > \ln a$ or

$$\left(\int_0^t \mathrm{tr}A(s)ds\right) > \ln a - \ln\left(|\det(\Psi(0))|\right)$$

It implies that $|\det(\Psi(t))|$ is bounded from below if and only if $\int_0^t \mathrm{tr}A(s)ds$ is bounded from below for $t \geq 0$ (cannot go to $-\infty$ with $t_k \to +\infty$ for some for some sequence of times $\{t_k\}_{k=1}^{\infty}$).

Solution to 9.

Abel's formula for fundamental matrix solution is $\det(\Psi(t)) = \det(\Psi(0))\exp\left(\int_0^t \mathrm{tr}A(s)ds\right)$. For

$\det(\exp(tA)) = \det(I)\exp\left(\int_0^t \mathrm{tr}A ds\right) = \exp(t\,\mathrm{tr}A)$

$\det((I+\varepsilon A)+O(\varepsilon^2)) = \det((I+\varepsilon A)+O(\varepsilon^2)-\exp(\varepsilon A)+\exp(\varepsilon A)) = \det(\exp(\varepsilon A)+O(\varepsilon^2)) = \det(\exp(\varepsilon\,\mathrm{tr}A)) + O(\varepsilon^2)$
$= \exp(\varepsilon\,\mathrm{tr}A) + O(\varepsilon^2) = 1 + \varepsilon\,\mathrm{tr}\,(A) + O(\varepsilon^2)$.

One can also give a direct proof considering an expansion of $\det((I+\varepsilon A)+O(\varepsilon^2))$ according to the addition rool for determinants and observing that the only terms of order zero and one in $\varepsilon \to 0$ in the determinant are 1 and $\varepsilon\, A_{ii}$. Adding the last ones leads to $\varepsilon\,\mathrm{tr}\,(A)$.

Solution to 10.

Consider the flow $\phi(t,x)$ corresponding to the autonomous equation $y' = f(y)$, $y \in \mathbb{R}^n$ mapping the domain Ω to the domain as $\Omega_t = \{y = \phi(t,x), x \in \Omega\}$ where y is the solution to the ODE $y' = f(y)$ with initial data $y(0) = x \in \Omega$.

Show that $\frac{d}{dt}(Vol(\Omega_t)) = \int_{\Omega_t} \mathrm{div}\,(f)\,dy$. **Hint:** use the result of Ex.9.

$(Vol(\Omega_t)) = \int_{\Omega_t} dx$

Considering derivative of the integral is useful to have the integration over a fixed domain and function under the integral depending on time. To implement this idea we introduce a change of variables such that the domain of integration for time t coinsides with the "initial" domain Ω_0 and

$(Vol(\Omega_t)) = \int_{\Omega_t} dx = \int_{\Omega_0} \left|\det\left[\frac{D\phi(t,x)}{Dx}\right]\right| dx$

Consider this integral for $t \to 0$.

$\frac{D}{Dx}\phi(t,x) = \frac{D}{Dx}\left[I\,x + t\,f(0,x) + O(t^2)\right] = \left[I + t\frac{D}{Dx}\,f(0,x) + O(t^2)\right]$, for $t \to 0$

$\det\left[\frac{D}{Dx}\phi(t,x)\right] = \det\left[I + t\frac{D}{Dx}\,f(0,x) + O(t^2)\right] = 1 + t\,\mathrm{tr}\left[\frac{D}{Dx}\,f(0,x)\right] + O(t^2) \geq 0$, for $t \to 0$

and $\left|\det\left[\frac{D\phi(t,x)}{Dx}\right]\right| = \det\left[\frac{D}{Dx}\phi(t,x)\right]$

$\mathrm{tr}\left[\frac{D}{Dx}\,f(0,x)\right] = div\,(f(0,x))$

$\frac{d}{dt}\left(Vol(\Omega_t)\right)_{t=0} = \int_{\Omega_0} div\,(f(0,x))\,dx$

The same argument works naturally for any time t.