1 LaSalle's invariance principle and it's applications to asymptotic stability and to ω - limit sets. §5.2

Example. An elementary introduction to LaSalle's invariance principle.

We like to investigate stability of equilibrium poin tin the origin for the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - x_2^3 \end{aligned}$$

Using the simple test function $V(x_1, x_2) = x_1^2 + x_2^2$ we observe that it is a Lyapunov function for the system:

$$V_f(x_1, x_2) = \nabla V \cdot f(x_1, x_2) = -2x_2^4 \leq 0$$

and the origin is a stable equilibrium point. On the other hand V is not a strong Lyapunov function, because $V_f(x_1, x_2) = 0$ not only in the origin, but on the whole x_1 - axis where x_2 is zero. We could ty to find a more sofisticated

On the other hand considering the vector field of velocities of this system on the x_1 - axis, we observe that they are crossing the x_1 - axis (even are orthogonal to it in this particular example) in all points except the origin. It means that all trajectories of the system cross and immediately leave the x_1 - axis that is the line where $V_f(x_1, x_2) = 0$ (the Lyapunov function is not strong). This observation shows that in fact the Lyapunov function $V(\varphi(t, \xi))$ is strictly monotone along trajectories $\varphi(t, \xi)$ everywhere except discret time moments, when $\varphi(t, \xi)$ crosses the x_1 - axis. More explicitely in polar coordinates r and θ:

$$\left(r^2\right)' = -2r^4 \sin^4 \theta$$

We can therefore conclude that $V(\varphi(t, \xi)) \searrow 0$ as $t \to \infty$ and therefore, the origin is asymptotically stable equilibrium of this system of equations.

One can also get a more explicit picture of this dynamics by looking on the equation for the polar angle θ:

$$\begin{aligned}\left(\frac{x_2}{x_1}\right)' &= (\tan(\theta))' = \frac{\theta'}{\cos^2(\theta)} \\ \frac{x_2' x_1 - x_1' x_2}{x_1^2} &= \frac{(-x_1 - x_2^3)\, x_1 - (x_2)\, x_2}{x_1^2} \\ &= \frac{(-x_1^2 - x_2^2 - x_1 x_2^3)}{x_1^2} = \frac{-r^2 - \cos\theta \sin^3\theta\, r^4}{r^2 \cos^2\theta}\end{aligned}$$

$$\begin{aligned}\theta' &= -1 - \cos\theta \sin^3\theta\, r^2 = -1 - \frac{\left(\sin 2\theta \sin^2\theta\right)\; r^2}{2} \\ &= -1 - \frac{\sin 2\theta(1 - \cos 2\theta) r^2}{4} < 0, \quad r < 2\end{aligned}$$

We see that the trajectories tend to the origin going (non-uniformly) as spirals clockwise around the origin.

This example demonstrates the main idea with applications of the LaSalles invariance principle to asymptotic stability of equilibrium points.

Proposition. Simple version of LaSalle's invariance principle. Theorem 5.15. p. 183

We find a simple "weak" Lyapunov function $V_f(z) \leq 0$ for $z \in U$ in the domain $U \subset G$, $0 \in U$. This fact implies stability of the equilibrium. Then we check what happens on the set $V_f^{-1}(0)$ where $V_f(z) = 0$. If the set $V_f^{-1}(0)$ contains no other orbits except the equilibrium point, this equilibrium point in the origin must be asymptotically stable and it's attracting region is the whole domain U where these properties are valid.

The next theprem gives a simple criterion for having the whole space as the domain of attraction for an asymptotically stable equilibrium point.

Theorem 5.22, p. 188. On global asymtotic stability

Assume that $G = \mathbb{R}^n$. Let the hypothesis of the orem 5.15 hold with $U = G = \mathbb{R}^n$. If in addition the Lyapunov function V is radially unbounded:

$$V(z) \to \infty, \quad \|z\| \to \infty$$

then the origin 0 is globally stable equilibrium that means that all solutions $\|\varphi(t,\xi)\| \to 0$, as $t \to \infty$.

Exercise.

Show that all trajectories of the system

$$\begin{aligned} x' &= y \\ y' &= -x - (1 - x^2)y \end{aligned}$$

that go through points in the domain $\|x\| < 1$, tend to the origin. Or by other words, show that the origin is an asymptotically stable equilibrium and that the circle $\|x\| < 1$ is it's domain of attraction.

Exercise 5.17

The aim of this exercise is to show that the condition of radial unboundedness in Theorem 5.22 is essential.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(z) = f(z_1, z_2) = \begin{cases} (-z_1, z_2) & \text{if } z_1^2 z_2^2 \geq 1 \\ (-z_1, 2z_1^2 z_2^3 - z_2) & \text{if } z_1^2 z_2^2 < 1 \,. \end{cases}$$

Define $V\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$V(z) = V(z_1, z_2) = z_1^2 + \frac{z_2^2}{1 + z_2^2} \,.$$

(a) Show that the equilibrium 0 of (5.1) is asymptotically stable.

(b) Show that the equilibrium 0 is *not* globally asymptotically stable.

(c) Show that V is not radially unbounded.

More general formulation and a proof of the LaSalle's invariance principle use some general properties of transition mappings, and ω - limit sets. We collect them here and give some comments about their proofs.

We consider I.V.P. and corresponding transition map $\varphi(t,\xi)$ for the system

$$\begin{aligned} x' &= f(x) \\ x(0) &= \xi \end{aligned}$$

with $f : G \to \mathbb{R}^n$, G - open, $G \subset \mathbb{R}^n$, f is locally Lipschitz, $\xi \in G$.

Proposition. Theorem 4.34, p.139 (consequence of Th. 4.29, p. 129)

The domain $D = \{(t,\xi) \in I_\xi \times G, \quad \xi \in G\}$ of the transition map $\varphi(t,\xi)$ is open and $\varphi(t,\xi)$ is continuous and even locally Lipschitz in D.

Proof of the Lipschitz property with respect to each of the variables follows from the integral form of the I.V.P. and for ξ variable - from an application of Grönwall inequality similar to the proof of uniqueness of solutions to I.V.P.

Proposition. Translation invariance of the transition mapping for autonomous systems

(a non-linear version of the Chapman-Kolmogorov relation) Theorem 4.35, p. 140.

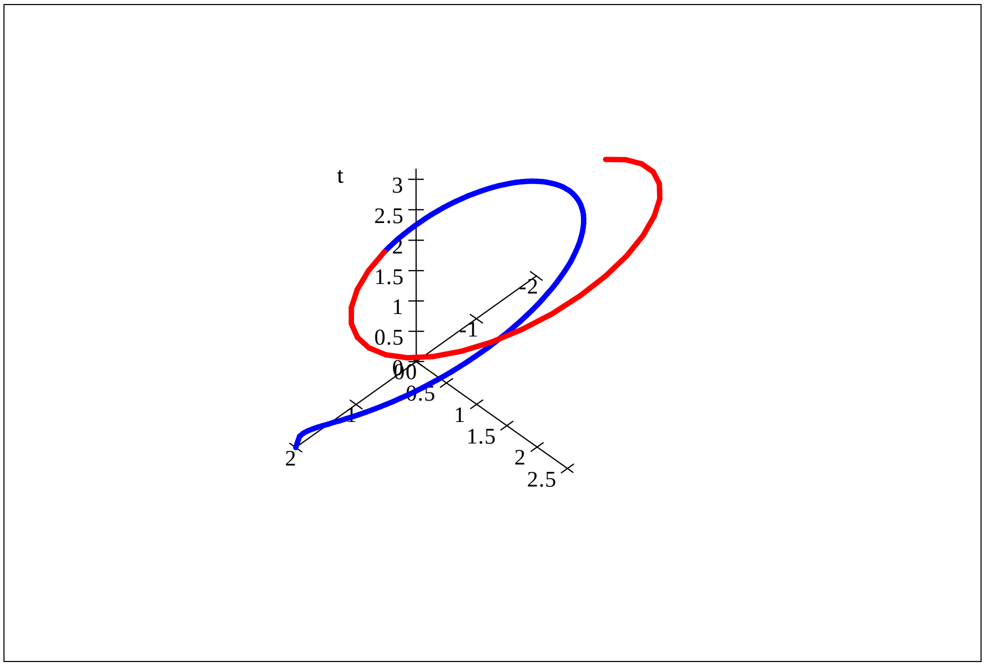
The transition mapping $\varphi(t,\xi)$ has properties

(1) $\varphi(0,\xi) = \xi$ for all $\xi \in G$

(2) if $\xi \in G$ and $\tau \in I_\xi = I_{\max}(\xi)$ - maximal interval for ξ, then

$$\begin{aligned} I_{\varphi(\tau,\xi)} &= I_\xi - \tau \\ \varphi(t+\tau,\xi) &= \varphi(t,\varphi(\tau,\xi)), \quad \forall t \in I_\xi - \tau \end{aligned}$$

Proof of this statement follows is similar to the proof of the Chapman Kolmogorov relations for linear systems.



We consider first a trajectory $\varphi(...,\xi)$ starting at the point $\xi \in G$ and finishing at time τ at the point $\varphi(\tau,\xi)$ (blue curve). Then we continue this movement from the last point $\varphi(\tau,\xi)$ during time t (red curve) coming finally to the point $\varphi(t,\varphi(\tau,\xi))$ in the right hand side of the equation in the conclusion. The fact that the equation is autonomous and independent of time makes that this movement is equivalent to just moving with the flow starting from the point ξ during the total time $t+\tau$, that is the left hand side in the equation. The illustration is borrowed from the proof for the linear systems. The only difference here is that we have a superposition $\varphi(t,\varphi(\tau,\xi))$ of transfer mappings in the non-linear case instead of the product of transfer matrices in the linear case (that is also a superposition for linear mappings).

Main theorem on the properties of limit sets.

The next theorem on the properties of ω - limit sets collects properties of ω - limit sets valid for systems

of any dimension, in contrast with the Poincare - Bendixson theorem and it's generalization, that give a description of ω - limit sets only for systems in plane, or on 2-dimensional manifolds.

Main theorem on the properties of limit sets. Theorem 4.38, p.143

We keep the same limitations and notations for the autonomous system as above.

Let $\xi \in G$. If the closure of the poistive semi-orbit $O^+(\xi)$ is compact and contained in G, then $\mathbb{R}_+ \subset I_\xi$ and the ω - limit set $\Omega(\xi) \subset G$ is

1) non-empty

2) compact

3) connected

4) invariant (both positively and negatively): for any ω - limit point $\eta \in \Omega(\xi)$, $I_\eta = \mathbb{R}$, and $\varphi(t, \eta) \in \Omega(\xi)$ for all $t \in \mathbb{R}$.

5) $\varphi(t, \xi)$ approaches $\Omega(\xi)$ as $t \to \infty$:

$$\lim_{t \to \infty} \operatorname{dist}(\varphi(t, \xi), \, \Omega(\xi)) = 0$$

Remark

The most interesting statement in the theorem is statement 4). It means that ω - limit sets consist of orbits of solutions to the system. Taking a starting point η on the limit set $\Omega(\xi)$ we get a trajectory $\varphi(t, \eta)$ that stays within this set $\Omega(\xi)$ infinitely long both in the future and in the past.

A simple tool to satisfy conditions in this theorem is to find a compact positively invariant set for the system that contains the point ξ. It can be done using one of two methods discussed earlier.

Proofs of statements in the theorem are based on: general properties of compact sets for 1) ,2), simple contradiction arguments and the definition of the limit sets for 3) and the translation property of the transition mapping $\varphi(t, \xi)$, together with continuity of $\varphi(t, \xi)$ for 4), a contradiction argument togehter with the definition of ω - limit sets.

LaSalle's invariance principle

We formulate now the LaSalle's invariance principle that generalizes ideas that we discussed in the introductory example and gives a handy instrument for localizing ω - limit sets of non-linear systems in arbitrary dimension.

Theorem 5.12, p.180

Assume that f is locally Lipschitz as before and let $\varphi(t,\xi)$ denote the flow generated by the corresponding system

$$x' = f(x)$$

Let $U \subset G$ be non-empty and open. Let $V : U \to \mathbb{R}$ be continuously differentiable and such that $V_f(z) = \nabla V \cdot f(z) \leq 0.$ for all $z \in U$. If $\xi \in U$ is such that the closure of the semi-orbit $O^+(\xi)$ is compact and is contained in U, then $\mathbb{R}_+ \subset I_\xi$ (maximal existence interval for ξ) and $\varphi(t,\xi)$ approaches as $t \to \infty$ the largest invariant set contained in $V_f^{-1}(0)$ that is the set where $V_f(z) = 0$.

Proof.

Proof given in the solution of Exercise 5.9, on p. 312.

Exercise 5.9

Set $x(\cdot) := \varphi(\cdot, \xi)$. By continuity of V and compactness of $\mathrm{cl}(O^+(\xi))$, V is bounded on $O^+(\xi)$ and so the function $V \circ x$ is bounded. Since $(\mathrm{d}/\mathrm{d}t)(V \circ x))(t) = V_f(x(t)) \leq 0$ for all $t \in \mathbb{R}_+$, $V \circ x$ is non-increasing. We conclude that the limit $\lim_{t\to\infty} V(x(t)) =: \lambda$ exists and is finite. Let $z \in \Omega(\xi)$ be arbitrary. Then there exists a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $x(t_n) \to z$ as $n \to \infty$. By continuity of V, it follows that $V(z) = \lambda$. Consequently,

$$V(z) = \lambda \quad \forall z \in \Omega(\xi). \tag{$*$}$$

By invariance of $\Omega(\xi)$, if $z \in \Omega(\xi)$, then $\varphi(t, z) \in \Omega(\xi)$ for all $t \in \mathbb{R}$ and so $V(\varphi(t, z)) = \lambda$ for all $t \in \mathbb{R}$. Therefore, $V_f(\varphi(t, z)) = 0$ for all $t \in \mathbb{R}$. Since $\varphi(0, z) = z$ and z is an arbitrary point of $\Omega(\xi)$, it follows that

$$V_f(z) = 0 \quad \forall\, z \in \Omega(\xi), \tag{$**$}$$

and so $\Omega(\xi) \subset V_f^{-1}(0)$. The claim now follows because, by Theorem 4.38, $\Omega(\xi)$ is invariant and $x(t)$ approaches $\Omega(\xi)$ as $t \to \infty$.

Comment. It might be tempting to conclude from $(*)$ that $(\nabla V)(z) = 0$ for all $z \in \Omega(\xi)$, which then immediately would yield $(**)$. However, this conclusion is not correct: the set $\Omega(\xi)$ is not open and therefore $(*)$ does not imply that $(\nabla V)(z) = 0$ for all $z \in \Omega(\xi)$. (The invalidity of the conclusion is illustrated by the following simple example: if $V(z) = \|z\|^2$ and $\Omega(\xi) = \{z \in \mathbb{R}^N : \|z\| = 1\}$, then $V(z) = 1$ for all $z \in \Omega(\xi)$, but $(\nabla V)(z) = 2z \neq 0$ for all $z \in \Omega(\xi)$.)

Example.

Consider the following system of ODEs: $\begin{cases} x' = & 2y \\ y' = & -x-(1-x^2)y \end{cases}$.

Show the asymptotic stability of the equilibrium point in the origin and find it's domain of attraction. **(4p)**

Solution.

We try the test function $V(x,y) = x^2 + 2y^2$ that leads to cancellation of mixed terms in the directional derivative along trajectories:

$V_f(x,y) = 4xy - 4xy - 4y^2(1-x^2) = -4y^2(1-x^2)$ that is not positive for $|x| \leq 1$. Therefore the origin is a stable stationary point. Checking the behavior of the system on the set of zeroes to $V_f(x,y)$ inside the stripe $|x| < 1$ we consider $(V_f)^{-1}(0) = \{(x,y) : y = 0, |x| < 1\}$. On this set $y' = -x$ and the only invariant set in $(V_f)^{-1}(0)$ is the origin. The LaSalles invariance principle implies that the origin is asymptotically stable and the domain of attraction is the largest set bounded by a level set of $V(x,y) = x^2 + 2y^2$ inside the stripe $|x| \leq 1$. The largest such set will be the interior of the ellipse $x^2 + 2y^2 = C$ such that is touches the lines $x = \pm 1$. Taking points $(\pm 1, 0)$ we conclude that $1 = C$. and the boundary of the domain of attraction is the ellipse $x^2+2y^2 = 1$ with halfs of axes 1 and $\sqrt{0.5}$: