Lecture notes on general and periodic linear ODE

Plan

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3. Uniqueness of solutions and dimenstion of solution space. Th. 2.5, p. 28, Prop. 2.7(1), p.30
4. Group properties of transition matrix function and transition mapping. Cor. 2.6, p. 29
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0.1 Transition matrix function, existence and equations.

The subject of this chapter of lecture notes is general non - autonomous linear systems of ODEs and in particular systems with periodic coefficients and Floquet theory for them.

The general theory for non - autonomous linear systems (systems with variable coeffcients) is very similar to one for systems with constant coefficients. The existence is established through the solution of the integral form of equations by iterations. Uniqueness is based on a general form of the Grönwall inequality that is also proved here in a very similar fasion. These results lead to the fundamental result on the dimension of the space of solutions that is based on the uniqueness result similarly to the proof for systems with constant coefficients. The essential difference from the case with constant coefficients is that in the case with variable coefficients one cannot find analytical solutions exept particular cases as systems with triangular matrices.

We consider the I.V.P. in the differential

$$x' = A(t)x(t), \quad x(\tau) = \xi \tag{1}$$

or in the integral form

$$x(t) = \xi + \int_{\tau}^{t} A(s)x(s)ds \tag{2}$$

with matrix valued function $A : J \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$) that is continuous or peacewise continuous on the interval J. The solution is defined as a continuous function $x(t)$ on an interval I that includes point τ acting into $\mathbb{R}^N$ or $\mathbb{C}^N$ satisfying the integral equation (2). By a version of Calculus main theorem (Newton-Leibnitz theorem) the solution defined in such a way will satisfy the differential equation (1) in points t where $A(t)$ is continuous.

We remind the following lemma considered in the beginning of the course.

Lemma. The set of solutions $\mathcal{S}_{\text{hom}}$ to (2) is a linear vector space.

It motivates us to search solution in the form $\Phi(t,s)\xi$ where $\Phi(t,s)$ is a continuous matrix valued function on $J \times J$ and ξ is an arbitrary initial data at $t = s : x(s) = \xi$. Substituting this expression into the integral form of the i.V.P. we arrive to the vector equation

$$\begin{aligned}\Phi(t,s)\xi &= \xi + \int_s^t A(\sigma)\Phi(\sigma,s)\xi d\sigma \Longrightarrow \\ \Phi(t,s)\xi &= \left(I + \int_s^t A(\sigma)\Phi(\sigma,s)d\sigma\right)\xi\end{aligned}$$

with arbitrary $\xi \in \mathbb{R}^N$ that implies the matrix equation for $\Phi(t,s)$:

$$\Phi(t,s) = I + \int_s^t A(\sigma)\Phi(\sigma,s)d\sigma \tag{3}$$

$$\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s); \quad \Phi(s,s) = I \tag{4}$$

We will solve this equation by means of iterational approximations $M_n(t,s)$ to $\Phi(t,s)$ introduced in the following way:

$$M_1(t,s) = I; \quad M_{n+1}(t,s) = I + \int_s^t A(\sigma)M_n(\sigma,s)d\sigma, \quad \forall \in \mathbb{N} \tag{5}$$

Lemma 2.1 , p. 24 and **Corollary 2.3,** p. 26 in L&R

For any closed and bounded interval $[a,b] \in J$ the sequence $\{M_n\}$ converges uniformly on $[a,b] \times [a,b]$ to a matrix valued function $\Phi(t,s)$ that satisfies the integral equation (3)

Proof.

The idea of the proof is instead of considering $M_n(t,s)$ to consider telescoping series with element $f_{n+1}(t,s) = M_{n+1}(t,s) - M_n(t,s)$, $f_1 = M_1 = I$ and with partial sumthat is equal to M_n:

$$M_n = \sum_{k=1}^n f_k$$

with $f_k(t,s)$ rewpresented as a repeated integral operator from (5):

$$f_{k+1}(t,s) = M_{k+1}(t,s) - M_k(t,s) = \int_s^t A(\sigma_1)\int_s^{\sigma_1} A(\sigma_2)...\int_s^{\sigma_{n-1}} A(\sigma_n)d\sigma_n...d\sigma_2 d\sigma_1$$

for all $(t,s) \in J \times J$, $\forall n \in \mathbb{N}$. Since $A(t)$ is piecewise continuous on J, it is bounded on any compact subinterval $[a, b]$:

$$\|A(t)\| \leq K \quad \forall t \in [a, b]$$

We observe that

$$\|f_n(t,s)\| = \|M_{n+1}(t,s) - M_n(t,s)\| \leq K^n \int_s^t \int_s^{\sigma_1} \dots \int_s^{\sigma_{n-1}} d\sigma_n \dots d\sigma_2 d\sigma_1$$

and after calculating the integral $\int_s^t \int_s^{\sigma_1} \dots \int_s^{\sigma_{n-1}} d\sigma_n \dots d\sigma_2 d\sigma_1 = \frac{1}{n!}(t-s)^n$

$$\|f_n(t,s)\| = \|M_{n+1}(t,s) - M_n(t,s)\| \leq \frac{K^n}{n!}(t-s)^n \leq \frac{K^n}{n!}(b-a)^n$$

The number series $\sum_{n=0}^{\infty} \frac{K^n}{n!}(b-a)^n$ is convergent to $\exp(K(b-a))$. Therefore by the Weierstrass criterion the series $\sum_{n=1}^{\infty} f_n(t,s)$ converges uniformly on $[a,b] \times [a,b]$ to the limit denoted here by $\Phi(t,s)$. It implies by construction, that the sequence $M_n(t,s)$ converges uniformly on $[a,b] \times [a,b]$ to the limit denoted here by $\Phi(t,s)$. Going to the limit in the relation defining iterations (5), we observe that the limit functional matrix $\Phi(t,s)$ satisfies the equation (3).■

Since the interval $[a, b] \in J$ is arbitrary we may define the function $\Phi : J = J \times J \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$) as the (pointwise) limit:

$$M_n(t,s) \to \Phi(t,s), \quad n \to \infty$$

Definition. The matrix $\Phi(t,\tau)$ is called transition matrix function.

Point out that $\Phi(t,t) = I$. The product $\Phi(t,\tau)\xi$ gives the solution to I.V.P. to the equation $x'(t) = A(t)x(t)$ with initial data $x(\tau) = \xi$.

0.2 Grönwall inequality. Uniqueness of solutions.

Gönwall 's lemma. Lemma 2.4., p. 27 in L&R. (we skipped it for now)

Let $I \subset \mathbb{R}$, be an interval, let $\tau \in I$, and let $g, h : I \to [0, \infty)$ be continuous nonnegative functions. If for some positive constant $c > 0$,

$$g(t) \leq c + \left| \int_\tau^t h(\sigma) g(\sigma) d\sigma \right| \quad \forall t \in I$$

then

$$g(t) \leq c \exp\left(\left| \int_\tau^t h(\sigma) d\sigma \right| \right) \quad \forall t \in I$$

Proof.

The proof uses the idea of integrating factor similar to the simpler case with constant $h = \|A\|$ considered before. Introduce $G, H : I \to [0, \infty)$ by

$$\begin{aligned} G(t) &= c + \left| \int_\tau^t h(\sigma) g(\sigma) d\sigma \right| \\ H(t) &= \left| \int_\tau^t h(\sigma) d\sigma \right| \end{aligned}$$

By hypothesis in the lemma, $0 \leq g(t) \leq G(t)$.We consider first the case $\tau < t$. Then the integrals in the expressions for G and H are nonnegative:

$$G(s) = c + \int_\tau^s h(\sigma) g(\sigma) d\sigma; \quad H(s) = \int_\tau^s h(\sigma) d\sigma, \quad \forall s \in [\tau, t]$$

Differentiation and Newton Leibnitz theorem imply

$$\begin{aligned} G'(s) &= h(s) g(s) \leq h(s) G(s) = H'(s) G(s), \quad \forall s \in [\tau, t] \\ G'(s) - H'(s) G(s) &\leq 0, \quad \forall s \in [\tau, t] \end{aligned}$$

Multiplying the inequality by $\exp(-H(s))$ and observing that

$$(G'(s) - H'(s) G(s)) \exp(-H(s)) = (G(s) \exp(-H(s)))'$$

we arrive to

$$(G(s) \exp(-H(s)))' \leq 0, \quad \forall s \in [\tau, t]$$

Integrating the last inequality from τ to t we arrive to

$$(G(t) \exp(-H(t))) \leq (G(\tau) \exp(-H(\tau))) = c$$

Therefore we arrive to the Grönwalls inequality:

$$(G(t)) \leq c \exp(H(t)) = c \exp\left(\int_\tau^t h(\sigma) d\sigma \right)$$

The case when $t < \tau$ is considered similarly by observing that for $t < \tau$

$$G(t) = c + \int_s^\tau h(\sigma)(\sigma) d\sigma; \quad H(t) = \int_s^\tau h(\sigma) d\sigma, \quad \forall s \in [t, \tau]$$

Do it as an exercise!

Uniqueness of solutions to I.V.P.

Theorem 2.5, p. 28 L&R

Let $(\tau, \xi) \in J \times \mathbb{R}^N (J \times \mathbb{C}^N)$. The function $x(t) = \Phi(t, \tau)\xi$ is a unique solution to the I.V.P. (1).If $y : J_y \to \mathbb{R}^N$ or $(\mathbb{C}^N)$ is a another solution to (1). then $y(t) = x(t)$ for all $t \in J_y$.

Proof.

The fact that $x(t) = \Phi(t,\tau)\xi$ is a solution to I.V:P. follows by construction and from the properties of the transition matrix function. Only uniqueness must be proved. Consider function $e(t) = x(t) - y(t)$ on the interval $J_y \subset J$. By linearity it satisfies the equation

$$e(t) = \int_\tau^t A(\sigma)e(\sigma)d\sigma, \quad \forall t \in J_y$$

Applying the triangle inequality for integrals we conslude that

$$\|e(t)\| \leq \int_\tau^t \|A(\sigma)\| \, \|e(\sigma)\| \, d\sigma, \quad \forall t \in J_y$$

point out that on a arbitrary bounded closed (compact) interval $[a,b] \subset J_y$ the peacewise continuous $A(\sigma)$ matrix valued function has a bounded norm $\|A(\sigma)\| < K$.Therefore for any $\tau, t \in [a,b]$

$$\|e(t)\| \leq \int_\tau^t K \, \|e(\sigma)\| \, d\sigma, \quad \forall t, \tau \in [a,b]$$

and by the simple variant Grönwall 's inequality $\|e(t)\| = 0$ for all $t \in [a,b]$ and therefore $y(t) = x(t)$ for all $t \in J_y$.

0.3 Solution space.

We have considered a particular variant of the following theorem in the case of linear systems of ODEs with constant coefficients. The formulation and the proof we suggested are based only on the fact that the set of solutions $\mathbb{S}_h$ is a linear vector space and on the property of uniqness of solutions. We repeat this argument here again with some corollaries about the structure of the transition matrix $\Phi(t,\tau)$.

Proposition 2.7 (1), p.30, L&R.

Let $b_1, ..., b_N$ be a basis in $\mathbb{R}^N$ (or $\mathbb{C}^N$) and let $\tau \in J$. Then the functions $y_j : J \to \mathbb{R}^N$ (or $\mathbb{C}^N$) defined as solutions $y_j(t) = \Phi(t,\tau)b_j$ with $j = 1, ..., N$ form a basis of the solution space $\mathbb{S}_h$. In particular $\mathbb{S}_h$ is N -dimensional and for every solution $x(t) : J \to \mathbb{R}^N$ (or $\mathbb{C}^N$) there exist scalars $\gamma_1, ...\gamma_N$ such that $x(t) = \sum_{j=1}^N \gamma_j y_j(t)$ for all $t \in J$.

Proof

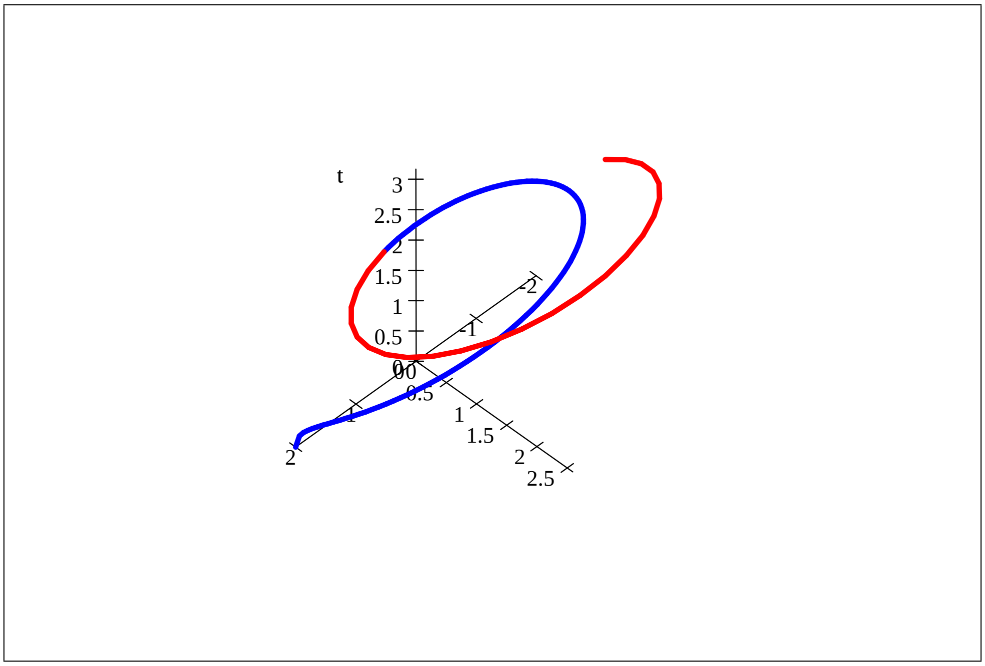
We can just repeat here the proof that we gave earlier. Point out that it is a bit more general comparing with one given in the book.

0.4 Group properties of transition matrix. Chapman - Kolmogorov relations.

The transition matrix $\Phi(t,\tau)$ defines a transition mapping $\varphi(t,\tau,\xi)$, that maps initial data ξ at time τ into the state $\varphi(t,\tau,\xi) = x(t) = \Phi(t,\tau)\xi$ of the system at time t.

Let us consider two consequtive solutions of the equation $x(t) = \Phi(t,\tau)\xi$ and $y(t) = \Phi(t,\sigma)\left(\Phi(\sigma,\tau)\xi\right)$ that continue each other in the time point $t = \sigma$ where the second solution $y(t)$ attains the initial state that is the point where the the first solution $x(t)$ arrives at time $t = \sigma$. Together with the uniquness of solutions this consideration leads to the group property of the transition mapping and transition matrix. The group property means that moving the system governed by the equation $x'(t) = A(t)x(t)$ from time τ to time t is the same as to move it first from time τ to time σ (blue curve) and then to move it without break from time σ to time t (red curve)

$$\Phi(t,\tau)\xi = \Phi(t,\sigma)\left[\Phi(\sigma,\tau)\xi\right]$$



Point out that these two "movements" do not need to go both in the positive direction in time as it is in the picture. One of these movements (or both) can go backward in time. Another observation is that the linearity of the system was not essential for this reasoning, only the uniqueness of solutions. We will use similar reasoning later for non-linear systems.

We have proven (almost) the following theorem.

Corollary 2.6, p.29 L&R (Chapman - Kolmogorov relations)

For all t, σ, $\tau \in J$

$$\begin{aligned} \Phi(t,\tau) &= \Phi(t,\sigma)\Phi(\sigma,\tau), \qquad (6) \\ \Phi(t,t) &= I, \\ \Phi(\tau,t) &= (\Phi(t,\tau))^{-1} \end{aligned}$$

Proof.

The first statement has been proven already. The second follows from the integral equation for the transfer matrix. The third one follows from the first two. We apply the first statement $\Phi(t,\tau)\ \Phi(\tau,t) = \Phi(t,t) = I$ therefore $\Phi(\tau,t)$ is the right inverse of $\Phi(t,\tau)$. The same argument for this expression with t and τ changed their roles leads to that $\Phi(\tau,t)$ is the left inverse of $\Phi(t,\tau)$.■

0.5 Fundamental matrix solution.

Introducing the transition matrix function $\Phi(t,\tau)$ for non-autonomous system of equations was similar to considering $\exp(At)$ for autonomous linear systems. We got a solution to an arbitrary I.V.P. by multiplying arbitrary initial data $x(\tau)=\xi$ with the the transition matrix function: $x(t)=\Phi(t,\tau)\xi$.

On the other hand we could construct a general solution to an autonomous linear system just by taking a linear combination of N linearly independent solutions to the system, because the dimension of the solution space is N.

The situation is exactly the same for non-autonomous linear systems with the difference that we in general cannot find a basis for the space of solutions analytically. It is possible only in some particular cases, for example for a triangular matrix $A(t)$.

Definition.

The functon $t\longmapsto\Psi(t)$ is called the fundamental matrix solution for the system $x'=A(t)x$, $x\in\mathbb{R}^n$ if it's columns $\Psi_k(t)$, $k=1,...,N$ are linearly independent solutions to the system (and therefore build a basis to the solution space): $\Psi_k'(t)=A(t)\Psi_k(t)$

It follows from the definition of the matrix product that

$$\Psi'(t)=A(t)\Psi(t)$$

General solution to the system is a linear combination of these vector valued functions: $x(t)=\Psi(t)C$ with an arbitrary constant vector $C\in\mathbb{R}^N$.

The fundamental matrix solution $\Psi(t)$ is an invertible matrix for all t because it's columns are linearly independent for all t.

There is a simple connection between $\Psi(t)$ and $\Phi(t,\tau)$.

Proposition 2.8 , p. 33

$$\Phi(t,\tau)=\Psi(t)\Psi^{-1}(\tau)$$

Proof.

The product $X(t,\tau)=\Psi(t)\Psi^{-1}(\tau)$ satisfies the equation

$$X'(t,\tau)=A(t)X(t,\tau)$$

in all points $t\in J$ where $A(t)$ is continuous, because each column in $\Psi(t)$ does it. On the other hand $\Psi(\tau)\Psi^{-1}(\tau)=I$. Therefore $X(t,\tau)=\Psi(t)\Psi^{-1}(\tau)$ satisfies the integral equation

$$X(t,\tau)=I+\int_\tau^t A(\sigma)X(\sigma,\tau)d\sigma$$

in all points $t\in J$ because each column in $\Psi(t)$ does it. The same equations are satisfied by $\Phi(t,\tau)$. By uniqueness of solutions to linear systems $\Phi(t,\tau)=X(t,\tau)=\Psi(t)\Psi^{-1}(\tau)$.

This proposition shows another way to calculate the transition matrix solution, because sometimes it is easier to find some basis for the space of solutions and to put it into a matrix $\Psi(t)$ instead of solving the matrix equation for $\Phi(t,\tau)$.

0.6 Non-homogeneous linear systems and Duhamel's formula in general case.

We consider the I.V.P. for non-homogeneous linear system

$$x'(t) = A(t)x(t) + b(t), \quad x(\tau) = \xi, \quad (\tau, \xi) \in J \times \mathbb{R}^N (J \times \mathbb{C}^N)$$

We suppose here that $A : J \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$) is continuous or peacewie continuous and denote by $\Phi(t,\tau)$ the transition matrix function generated by $A(t)$. We rewrite the I.V.P. for the system also in integral form

$$x'(t) = \xi + \int_\tau^t \left(A(\sigma)x(\sigma) + b(\sigma)\right) d\sigma,$$

that allows to consider continuous solutions in the case when A is only peacewie continuous. In this case solutions satisfy the differential form of the problem in time points outside of discontinuties of A.

Theorem 2.15, p. 41 L&R

Let $(\tau, \xi) \in J \times \mathbb{R}^N$.The function

$$x(t) = \Phi(t,\tau)\xi + \int_\tau^t \Phi(t,\sigma)b(\sigma)d\sigma,$$

is a unique solution to the I.V.P. above.

Proof. A siimpler proof can be given for points t outside the discontinuties of A.

Apply the Chapman-Kolmogorov relation to the transition matrix under the integral: $\Phi(t,\sigma) = \Phi(t,0)\Phi(0,\sigma)$ and calculate derivative of the integral in the expression for the solution.

$$\begin{aligned}
& \frac{d}{dt}\left(\int_\tau^t \Phi(t,\sigma)b(\sigma)d\sigma\right) \\
= \ & \frac{d}{dt}\left(\int_\tau^t \Phi(t,0)\Phi(0,\sigma)b(\sigma)d\sigma\right) = \frac{d}{dt}\left(\Phi(t,0)\int_\tau^t \Phi(0,\sigma)b(\sigma)d\sigma\right) \\
= \ & \left(\frac{d}{dt}\Phi(t,0)\right)\int_\tau^t \Phi(0,\sigma)b(\sigma)d\sigma + \left(\Phi(t,0)\frac{d}{dt}\left(\int_\tau^t \Phi(0,\sigma)b(\sigma)d\sigma\right)\right) \\
= \ & A\Phi(t,0)\int_\tau^t \Phi(0,\sigma)b(\sigma)d\sigma + \Phi(t,0)\Phi(0,t)b(\sigma)
\end{aligned}$$

Observe that by Chapman -Kolmogorov relation $\Phi(t,0)\Phi(0,t) = \Phi(t,t) = I$, and $\Phi(t,0)\Phi(0,\sigma) = \Phi(t,\sigma)$. It implies simplifications in the last formula and finally

$$\frac{d}{dt}\left(\int_\tau^t \Phi(t,\sigma)b(\sigma)d\sigma\right) = A\left(\int_\tau^t \Phi(t,\sigma)b(\sigma)d\sigma\right) + b(t)$$

Therefore $\int_\tau^t \Phi(t,\sigma)b(\sigma)d\sigma$ is the solution to the inhomogeneous equation with initial condition zero. Together with the solution $\Phi(t,\tau)\xi$ to the homogeneous equation, satisfying the initial condition $\Phi(\tau,\tau)\xi = \xi$ we conclude that $x(t) = \Phi(t,\tau)\xi + \int_\tau^t \Phi(t,\sigma)b(\sigma)d\sigma$, is a solution to the I.V.P. above. The uniqueness follows if we consider difference between two solutions $x(t)$ and $y(t)$ with the same initial condition: $z(t) = x(t) - y(t)$ that evidently satisfies the homogeneous equation $z'(t) = A(t)z(t)$ and the zero initial condition $z(\tau) = 0$. The known result for homogeneous linear systems based on Grönwall's inequality implies that $z(t) = 0$ on J.

Another proof based on the integral formulation of the problem and on the explicit checking that $x(t)$ satisfies it, is given in the book on page 41.■

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0.7 Systems with periodic coefficients: Floquet theory

We consider here linear homogeneous systems of ODE's with $J = R$ and a continuous or piecewise continuous matrix $A : \mathbb{R} \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$), with period $p > 0$:

$$x'(t) = A(t)x(t), \quad A(t+p) = A(t), \quad \forall t \in \mathbb{R}$$

Let Φ be a transition function generated by a påeriodic $A(t)$.

We are going to prove an important shift invariance property of this transition matrix function, namely that

$$\Phi(t+p, \tau+p) = \Phi(t,\tau) \tag{7}$$

Another property specifying further how the periodicity of the system influences properties of solutions.

$$\begin{aligned} \Phi(t+p,\tau) &= \Phi(t,0)\Phi(p,0)\Phi(0,\tau) \\ \Phi(t+n\,p,\tau) &= \Phi(t,0)\left[\Phi(p,0)\right]^n \Phi(0,\tau) \end{aligned} \tag{8}$$

for any $(t,\tau) \in \mathbb{R}\times\mathbb{R}$.

Definition of the Monodromy matrix

The matrix $\Phi(p,0)$ for a periodic linear system with period p is called the monodromy matrix.

Proof.

This first property is untuitively clear. The matrix $\Phi(t,\tau)$ satisfies the equation $\frac{\partial}{\partial t}\Phi(t,\tau) = A(t)\Phi(t,\tau)$ with initial condition , $\Phi(t,\tau)|_{t=\tau} = I$ and the matrix $\Phi(t+p,\tau+p)$ satisfies the equation $\frac{\partial}{\partial t}\Phi(t+p,\tau+p) = A(t+p)\Phi(t+p,\tau+p)$ with initial condition , $\Phi(t+p,\tau+p)|_{t=\tau} = I$.

Now we observe that $A(t) = A(t+p)$. Substituting it in the second equation we get the equation $\frac{\partial}{\partial t}\Phi(t+p,\tau+p) = A(t)\Phi(t+p,\tau+p)$ with the same initial condition, $\Phi(\tau+p,\tau+p) = I$ on the interval $t \in [\tau, t)$.

It implies that $\Phi(t,\tau)$ and $\Phi(t+p,\tau+p)$ satisfy in fact the same equation with the same initial conditions $\Phi(t+p,\tau+p)|_{t=\tau} = I$. The uniqueness of solutions implies that they must be equal: $\Phi(t+p,\tau+p) = \Phi(t,\tau)$.

A prove using the integral form of the equation is presented in the course book.■

The second property is proven by a combination of the shifting property with the Chapman-Kolmogorov relasions.

$$\begin{aligned}
&\Phi(t+p,\tau) \overset{Ch.-Kol.}{=} \Phi(t+p,\tau+p)\Phi(\tau+p,\tau) \overset{Shift}{=} \Phi(t,\tau)\Phi(\tau,\tau-p) \\
&\overset{Ch.-Kol.}{=} \Phi(t,\tau)\Phi(\tau,0)\Phi(0,\tau-p) \overset{Ch.-Kol.}{=} \Phi(t,0)\Phi(p,\tau) \\
&\overset{Ch.-Kol.}{=} \Phi(t,0)\Phi(p,0)\Phi(0,\tau)
\end{aligned}$$

another equality for shift $n\,p$ in time is derived by the repetition of the last argument.

■

The main idea of the Floquet theory.

The monodromy matrix $\Phi(p,0)$ is a particular transition matrix that maps initial data at time $\tau = 0$ to the state of the system after one period. A particular property of this matrix in the case of periodic systems is that similar mapping to the sate after time equal to n periods is just

$$\Phi(n \cdot p, 0) = [\Phi(p,0)]^n$$

This property is similar to properties of autonomous linear systems where $\Phi(t,0) = \exp(At)$ and therefore

$$\Phi(n \cdot p, 0) = \exp(A(n \cdot p)) = [\exp(A(p))]^n = [\Phi(\,p,0)]^n \tag{9}$$

that follows from the factorisation property of the exponent of two commuting matrices:

$$\exp(A+B) = \exp(A)\exp(B)$$

In the case of periodic systems this factorisation applies only for shifts on time that is integer number of periods.

But it is still a remarcable property that behaviour of solutions is described by a repeated multiplication by a constant matrix in certain time points: p, $2p$, $3p$, ...:

$$x'(t) = A(t)x(t), \quad x(0) = \xi.$$

$$x(np) = [\Phi(\,p,0)]^n\,\xi\,, \quad n = 0,1,2,...$$

The first idea of the Floquet theory is to represent $x(np)$ at times $t = np$ similarly as for autonomous systems, namely with the help of an exponent of some constant matrix F times the time argument: $t = np$.

$$x(np) = [\Phi(\,p,0)]^n\,\xi = \exp(np\,F)\xi = [\exp(p\,F)]^n\,\xi$$

It means that the matrix F in such representation must satisfy the relation

$$\Phi(\,p,0) = \exp(pF).$$

Therefore the matrix pF must be something like the logarithm of the monodromy matrix:

$$pF = \log(\Phi(\,p, 0))$$

Definition. A matrix $G \in \mathbb{C}^{N\times N}$ is called a loragithm of the matrix $H \in \mathbb{C}^{N\times N}$ if

$$H = \exp(G)$$

We write in this case $G = \log(H)$.

We are going to prove soon that for any non-singular matrix H there is a logarithml $\log(H)$ in this sense. Point out that the monodromy matrix $\Phi(\,p, 0)$ is always non-singular, because columns in a transfer matrix $\Phi(\,t, 0)$ are always linearly independent.

The logarithm of a matrix is not uniquely defined in the same way as it is not unique for complex and real numbers z:

$$\ln(z) = \ln(|z|) + i\arg(z) \tag{10}$$

because the argument $\arg(z)$ of a complex number is defined only up to $2\pi k$, $k = \pm 1, \pm 2, ...$

One can choose a unique branch for the logarithm function, called the *principle logarithm* or $\mathrm{Log}\,(z)$ by choosing the argument in the last formula (10) only in the interval $[0, 2\pi)$.

We will suspend the discussion of matrix logarithm now and will consider first an application of it to the analysis of solutions to periodic linear systems of ODEs.

The main idea in the Floquet theory is the "approximation" of the transfer matrix $\Phi(\,t, 0)$ for a periodic linear system with matrix $A(t) = A(p + t)$ by the transfer matrix $\exp\,(t\,F)$ for an autonomous system

$$y'(t) = [F]\,y(t)$$

with the constant matrix $F = \left[\frac{1}{p}G\right]$ where

$$G = \log(\Phi(\,p, 0)) \tag{11}$$

These two transfer matrices coinside in points $t = 0, p, 2p, 3p, ...$

$$\Phi(\,np, 0) = \exp\left((np)\,[F]\right) \tag{12}$$

The deviation of $\Phi(\,t, 0)$ from $\exp\,(t\,F)$ in intermediate points within one period can be expressed by a factor $\Theta(t)$ so that

$$\Phi(\,t, 0) = \Theta(t)\exp\,(tF)$$

The matrix function $\Theta(t)$ must be equal to the unit matrix I in the points $t = 0, p, 2p, ...$ because in these points these two transfer functions coinside by construction, see (12).

The exact formulation of the properties of such factorisation is given in the following Theorem by Floquet.

Theorem 2.30 , p. 53. Floquet theorem

Let $G \in \mathbb{C}^{N\times N}$ be a logarithm of the monodromy matrix $\Phi(p,0)$. There exists a periodic piecewise continuously differentiable function $\Theta(t) : \mathbb{R} \to \mathbb{C}^{N\times N}$, with $\Theta(0) = I$ and $\Theta(t)$ non-singular (invertible, all eigenvalues are non-zero) for all t, such that

$$\Phi(t,0) = \Theta(t)\exp\left(\frac{t}{p}G\right), \quad \forall t \in \mathbb{R} \tag{13}$$

Proof.

We remind the main property (8) of the monodromy matrix:

$$\Phi(t+p,0) = \Phi(t+p,p)\Phi(p,0) = \Phi(t,0)\Phi(p,0)$$

where we applied first the Chapmen Kolmogorov relation (6) and then the shift invariance (7) of the transfer function $\Phi(t,\tau)$. to a periodic linear system

We denote $\frac{1}{p}G$ by F for convenience, so that $G = pF$, and define the function $\Theta(t)$ after the desired relation (13)

$$\Theta(t) = \Phi(t,0)\exp\left(-\frac{t}{p}G\right) = \Phi(t,0)\exp(-tF)$$

The function $\Theta(t)$ is well defined in such a way. The problem is to show that it has desired properties: p - periodicity and satisfies initial conditions.

We remind that $\Theta(0) = I$ and even $\Theta(np) = I$ for all $n = 0, 1, 2, 3, ...$

$\Phi(t,0)$ is peacewise contiuously differentiable or contiuously differentiable depending on if $A(t)$ is peacewise continuous or continuous. Therefore $\Theta(t)$ has the same property because $\exp\left(-\frac{t}{p}G\right)$ is continosly differentiable. $\Theta(t)$ is also invertible for all t as a product of two invertible matrices $\Phi(t,0)$ and $\exp(-tF)$.

We check now that $\Theta(t)$ is p - periodic, namely that $\Theta(t+p) = \Theta(t)$ for all $t \in \mathbb{R}$.

$$\begin{aligned}
\Theta(t+p) &= \Phi(t+p,0)\exp(-(t+p)F) \\
&= \Phi(t+p,0)\exp(-pF)\exp(-tF) = \Phi(t+p,0)\exp(-G)\exp(-tF)
\end{aligned}$$

We remind that $\exp(G) = \Phi(p,0)$, therefore $\exp(-G) = (\exp(G))^{-1} = \Phi(p,0)^{-1} = \Phi(0,p)$. Therefore, using relation (8) for $\Phi(t+p,0) = \Phi(t,0)\Phi(p,0)$ together with the relation $\exp(-G) = \Phi(0,p)$, we arrive to

$$\Theta(t+p) = \Phi(t,0)\Phi(p,0)\Phi(0,p)\exp(-tF) = \Phi(t,0)\exp(-tF) \stackrel{\text{def}}{=} \Theta(t),$$

where we also used that $\Phi(p,0)\Phi(0,p) = I$ in the last step. Therefore $\Theta(t)$ is periodic with period p.■

0.8 Logarithm of a matrix. Existence and calculation.

We will formulate a theorem and give a proof to it (simpler than in the book) about the existence of a matrix logarithm.

Consider a nonsingular matrix H and it's a canonical Jordan form J: $H = TJT^{-1}$where T is invertible matrix. Then if there is $Q \in \mathbb{C}^{N\times N}$, such that $\exp(Q) = J$ that means

$$Q = \log(J),$$

then according to the properties of the exponent of similar matrices, and the definition of matrix logarithm

$$H = T\exp(Q)T^{-1} = \exp\left(TQT^{-1}\right) \overset{\text{def}}{=} \exp(\log(H))$$

It means that to calculate logarithm of an arbitrary matrix H it is enough to calculate the logarithm of it's Jordan canonical form. For $H = TJT^{-1}$

$$\log(H) = T\log(J)T^{-1}$$

Definition.

We say that G is a principal logarithm of the matrix H if G is a matrix logarithm of H and

$$\begin{aligned} \sigma(H) &= \{\exp(\lambda) : \quad \lambda \in \sigma(G)\} \\ \sigma(G) &= \{\mathrm{Log}(\lambda) : \quad \lambda \in \sigma(H)\} \end{aligned}$$

where $\mathrm{Log}(\lambda)$ is the scalar principal logarithm $z = e^{\mathrm{Log}(z)}$ and $\mathrm{Im}(\mathrm{Log}\,(z)) \in [0, 2\pi)$. This definition implies the explicit one to one correspondence between eigenvalues to H and eigenvalues to G. Essentially the second relation is non-trivial.

Theorem.Proposition 2.29, p. 53.

If $H \in \mathbb{C}^{N\times N}$ is invertible, then there exists a principle logarithm $\mathrm{Log}\,(H)$.

Proof.

We have established above that it is enough to investigate existence of logarithm for the similar canonical Jordan form J of the matrix. So without loss of generality we may assume that H is canonical Jordan form J. Therefore it is enough to establish the existence of logarithm for each Jordan block J_j in J, $j = 1, ..., s$ where s is the number of distinct eigenvectors to H and J_j has size $n_j \times n_j$

$$J_j = \begin{bmatrix} \lambda_j & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda_j & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & \lambda_j & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda_j \end{bmatrix}$$

$J_j = \lambda_j \left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right)$ where

$$\mathcal{N}_j = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

From the classical Maclaurin series for $\log(1+x) = \sum_{p=1}^{\infty} \frac{-(-1)^{p+1}}{p} x^p$ valid for $|x| < 1$, and for exp we get

$$\exp(\log(1+x)) = 1 + x$$

We formally write the Maclaurin series for $\log(1 + \frac{1}{\lambda_j}\mathcal{N}_j)$:

$$\log\left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right) = \sum_{p=1}^{n_j-1} \frac{(-1)^{p+1}}{p} \left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p$$

and observe that the Maclaurin series for $\log(1 + \frac{1}{\lambda_j}\mathcal{N}_j)$ is a finite sum because all larger powers of $\mathcal{N}_j$ in the series cancel. We have therefore that

$$\exp\left(\log\left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right)\right) = I + \frac{1}{\lambda_j}\mathcal{N}_j$$

We define

$$G_j \stackrel{\text{def}}{=} \log(\lambda_j)I + \sum_{p=1}^{n_j-1} \frac{(-1)^{p+1}}{p} \left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p$$

Then we check that this expression G_j is actually a matrix logarithm $\log(J_j)$ for the Jordan block J_j by checking that is satisfies the definition of the matrix logarithm. Point out that the diagonal matrix $\log(\lambda_j)I$ commutes with any matrix. Therefore

$$\begin{aligned} \exp(G_j) &= \exp\left(\log(\lambda_j)I + \sum_{p=1}^{n_j-1} \frac{(-1)^{p+1}}{p} \left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p\right) \\ &= \exp\left(\log(\lambda_j)I\right) \exp\left(\sum_{p=1}^{n_j-1} \frac{(-1)^{p+1}}{p} \left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p\right) \\ &= \exp\left(\log(\lambda_j)I\right) \exp\left(\log\left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right)\right) = \lambda_j \left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right) = J_j \end{aligned}$$

All calculations that we carried out are correct because $\lambda_j \neq 0$. We can choose logarithms $\log(\lambda_j)$ in these calculations as principle values of logarithm $\mathrm{Log}(\lambda_j)$. In this case the logarithm of J_j will be principal logarith, because there will be one to one correspondence between eigenvalues λ_j to J_j and eigenvalues $\mathrm{Log}(\lambda_j)$ to $\mathrm{Log}(J_j)$ that are that are diagonal elements in correpsonding matrices. They will have the same algebraic multiplicity and the same geometric multiplicity 1 (one linearly independent eigenvector for each Jordan block)

Therefore the existense of the principal logarithm is established also for J and for H, that is a matrix similar to J. The same correspondence as above is valid for the eigenvalues to H and to $\mathrm{Log}(H)$ because characteristic eigenvalues to similar matrices H and J are the same. The number of linearly independent eigenvectors corresponding to each distinct eigenvalue (geometric multiplicity) will be also the same.

In the Jordan canonical form J eigenvalues stand on diagonal and are easy to control.

0.9 Floquet multiplicators and exponents and bounds of solutions to periodic systems. equations.

Definition.

Eigenvalues of the monodromy matrix $\Phi(p,0)$ are called Floquet's multipliers or characteristic mutipliers.

A Floquet multiplier is called semisimple if it is semisimple as an eigenvalue to the monodromy matrix $\Phi(p,0)$.

Definition.

Eigenvalues of the logarithm of the monodromy matrix are called Floquet's exponents or characteristic exponents.

Theorem 2.31, p.54 on boundedness and zero limits of solutions to periodic linear systems.

1) Every solution to a periodic linear system is bounded on R_+ if and only if the abosolute value of each Floquet multiplier is not greater than 1 and any Floquet multiplier with absolute value 1 is semisimple.

2) Every solution to a periodic linear system tends to zero at $t \to \infty$ if and inly if the absolute value of each Floquet multiplier is strictly less than 1.

By Floquet theorem any solution $x(t)$ to system

$$x'(t) = A(t)x(t), \quad A(t+p) = A(t), \forall t \in \mathbb{R} \tag{14}$$

satisfying initial conditions $x(\tau) = \xi$, is represented as

$$x(t) = \Phi(t,\tau)\xi = \Theta(t)\exp(tF)\Phi(0,\tau)\xi = \Theta(t)\exp(tF)\zeta$$

where $F = \frac{1}{p}\mathrm{Log}(\Phi(p,0)), \Phi(0,\tau)\xi = \zeta$. $\Theta(t)$ is a p - periodic continuous or piecewise continuous matrix valued function. $\Theta(t)$ is invertible for all t.

We define $y(t) = \exp(tF)\zeta$ as a solution to the equation

$$y'(t) = F\,y, \quad y(0) = \zeta \tag{15}$$

$y(t) = \Theta^{-1}(t)x(t)$, and $x(t) = \Theta(t)y(t)$. The mapping $\Theta(t)$ determines a one to one correspondence between solutions to the periodic system (14) and the autonomous system (15).The periodicity and continuity properties of $\Theta(t)$ and $\Theta^{-1}(t)$ imply that there is a constant $M > 0$ such that $\|\Theta(t)\| \leq M$ and $\|\Theta^{-1}(t)\| \leq M$ for all $t \in \mathbb{R}$. It implies that $\|x(t)\| \leq M\,\|y(t)\|$ and $\|y(t)\| \leq M\,\|x(t)\|$.

Therefore

1)$\|x(t)\|$ is bounded on $\mathbb{R}_+$ if and only if corresponding $\|y(t)\| = \|\exp(tF)\zeta\|$ is bounded on $\mathbb{R}_+$.

2) $\|x(t)\| \to 0$ when $t \to \infty$ if and only if corresponding $\|y(t)\| \to 0$ when $t \to \infty$.

Since $\operatorname{Log}(\Phi(p,0)) = G = pF$, it follows that

$$\begin{aligned}
\sigma(\Phi(p,0)) &= \{\exp(\lambda p): \quad \lambda \in \sigma(F)\} \\
\sigma(F) &= \left\{\frac{1}{p}\operatorname{Log}(\mu) : \mu \in \sigma(\Phi(p,0))\right\}
\end{aligned}$$

and that algebraic and geometric multiplicities of each $\lambda \in \sigma(F)$ coinside with those of $\exp(p\lambda) \in \sigma(\Phi(p,0))$ .We use now that

$$\begin{aligned}
\operatorname{Log}(z) &= \ln(|z|) + i\arg(z) \\
\exp(z) &= \exp(\operatorname{Re} z)(\cos(\operatorname{Im} z) + i\sin(\operatorname{Im} z)
\end{aligned}$$

The following connections between properties of Floquet multipliers and propertis of corresponding eigenvalues to the matrix $F = \frac{1}{p}\operatorname{Log}(\Phi(p,0))$ are a direct consequence:

a) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$,has$|\mu| < 1$ if and only if $\operatorname{Re}\operatorname{Log}(\mu) < 0$ that is if the corresponding eigenvalue $\lambda = \frac{1}{p}\operatorname{Log}(\mu)$ to F has $\operatorname{Re}\operatorname{Log}(\mu) < 0$.

b) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$,has $|\mu| \leq 1$ if and only if $\operatorname{Re}\operatorname{Log}(\mu) \leq 0$ that is if the corresponding eigenvalue $\lambda = \frac{1}{p}\operatorname{Log}(\mu)$ to F has $\operatorname{Re}\operatorname{Log}(\mu) \leq 0$.

c) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$, with $|\mu| = 1$ is semisimple if and only if the corresponding eigenvalue $\lambda = \frac{1}{p}\operatorname{Log}(\mu)$ to F has $\operatorname{Re}\operatorname{Log}(\mu) = 0$ is semisimple.

Known relations between properties of solutions to an autonomous system and the spectrum of corresponding matrix applied to the system $y'(t) = F\,y$ and to the spectrum $\sigma(F)$ of the matrix F together with statements 1),2), a),b),c) in the present proof imply the statement of the theorem.■

Example.

Consider the following scalar linear equation with periodic coefficient $A(t) = (\sin(4t) - 0.1)$ with period $p = 0.5\pi$:

$$\frac{dx}{dt} = (\sin(4t) - 0.1)\,x,$$

We will find the monodromy matrix for this simple equation and demonstrate all objects related to the Floquet theorem.

The exact general solution is: $x(t) = C\exp(-0.25\cos(4.0t))\,e^{-0.1t}$ with arbitrary constant C can be found by the method with integrating factor.

To find the solution equal to 1 at $t = 0$ that is the transfer "matrix" in the scalar case, we calculate the expression $\exp(-0.25\cos(4.0t))\,e^{-0.1t}\big|_{t=0} = 0.7788$ and choose $C = \frac{1}{0.7788}$ in the expression for the general solution $x(t)$.

The transfer "matrix" is: $\Phi(t,0) = \frac{1}{0.7788}\exp(-0.25\cos(4.0t))\,e^{-0.1t}$

The period of the coefficient in the system is $p = 0.5\pi$ and the monodromy matrix is $\Phi(0.5\pi, 0)$:

$\Phi(p,0) = \frac{1}{0.7788}\exp(-0.25\cos(4.0t))\,e^{-0.1t}\big|_{t=0.5\pi} = 0.85464$

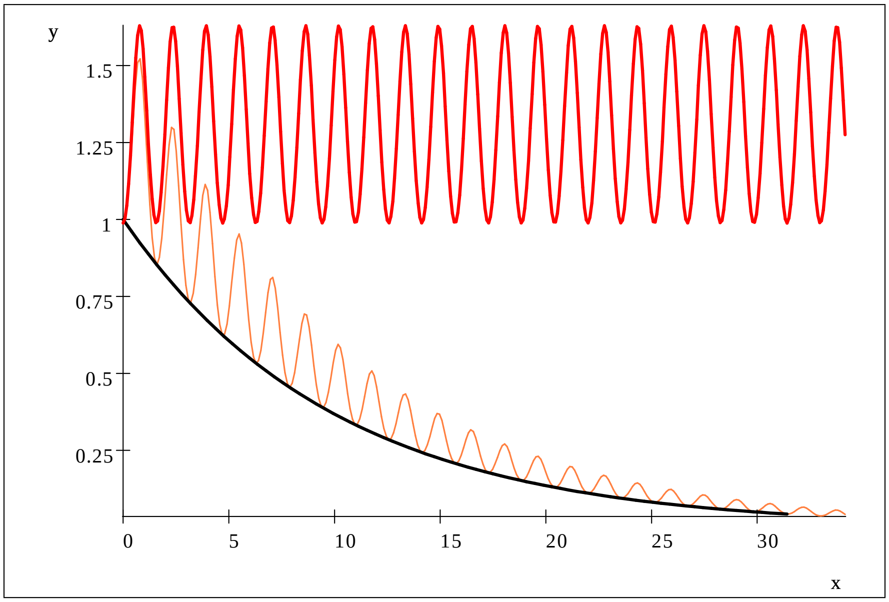
The eigenvalue μ of the (1x1) "monodromy matrix" $\Phi(p,0)$ coinsides with it's value: $\mu = 0.85464 < 1$ and is strictly less than 1.

$G = \ln\left(\Phi(p,0)\right) = \ln(\frac{1}{0.7788}\exp\left(-0.25\cos\left(4.0t\right)\right)e^{-0.1t})\big|_{t=0.5\pi} = -0.15708$
$F = \frac{G}{p} = \frac{-0.15708}{0.5\pi} = -0.1 < 0$

Therefore the eigenvalue $\lambda = -0.1$ of the "matrix" $F = \frac{1}{p}G$ is negative.

The transfer matrix to the system $y'(1) = Fy(t)$ is $\exp(t\frac{G}{p}) = \exp(-0.1t)$.

Compare black and green grafs for $\exp(t\frac{G}{p})$ and for $\Phi(t,0) = \frac{1}{0.7788}\exp\left(-0.25\cos\left(4.0t\right)\right)e^{-0.1t}$ and introducing a "corrector" multiplicator $\Theta(t)$ by $\Phi(t,0) = \Theta(t)\exp(t\frac{G}{p})$ observe that $\Theta(t) = \frac{1}{0.7788}\exp\left(-0.25\cos\left(4.0t\right)\right)$ is a $p = 0.5\pi$ - periodic function equal to 1 in all points $t = pn = (0.5\pi)\,n$, $n = 1, 2, 3...$(red graf).



Proposition 2.20. On periodic solutions to periodic linear systems

The system $x'(t) = A(t)x(t)$ with p - periodic $A(t) = A(t+p)$ has a non-zero p - periodic solution if and only if the monodromy matrix $\Phi(p,0)$ has an eigenvalue $\lambda = 1$. A more general statement is also valid.

The system has a non-zero $n\,p$ - periodic solution for $n \in \mathbb{N}$ if and only if the monodromy matrix $\Phi(p,0)$ has an eigenvalue λ such that $\lambda^n = 1$.

Proof. Consider an eigenvector v corresponding to the eigenvalue λ. Then $v \neq 0$, $\Phi(p,0)v = \lambda v$ and

$$[\Phi(p,0)]^n\,v = \lambda^n v = v$$

We will show that the solution to the system, with initial data $x(0) = v$ has period np. This solution is given transition matrix: $x(t) = \Phi(t,0)v$. Using this representation and applying the factorisation property of transition matrices for periodic systems we arrive to

$$x(t+np) = \Phi(t+np,0)v = \Phi(t,0)\,[\Phi(p,0)]^n\,v = \Phi(t,0)v = x(t), \quad \forall t \in \mathbb{R}$$

It shows that $x(t)$ is periodic with period $n\,p$.

Supposting that there is a periodic solution $x(t+np) = x(t)$ and repeating the same calculation backwards we arrive that $x(0) = v$ is an eigenvalue corresponding to an eigenvalue λ such that $\lambda^n = 1$.

Carry out this backward argument as an exercise!

■

0.10 Abel - Liouville's formula.

Lemma about the derivative of a determinant of a matrix valued function.

Let $B : J \to \mathbb{R}^{N\times N}$ be differentiable. Then the derivative of it's determinant satisfies the following formula

$$(\det(B(t))' = \sum_{k=1}^{N} \det\left(U_k(B)\right)$$

where matrices $U_k(B)$ have the same columns $b_k(t)$ as the matrix $B(t) = [b_1(t), ..., b_N(t)]$ except the k -th column exchanged by the column of derivatives of the k-th column in $B(t)$.

$$U_k(B) = \left[b_1(t), ..., \left[\frac{d}{dt}b_k(t)\right], ..., b_N(t)\right]$$

A similar relation can be written for rows instead of columns.

An elementary proof can be carried out using the definition of derivative as a limit of a finite difference and repeated application of the addition formula for determinants. **Prove it as an exercise on properties of determinants!**

Consider a homogeneous linear system of ODEs $x'(t) = A(t)x(t)$ and N solutions $y_1(t), y_2(t),...,y_N(t)$ to it. Consider the matrix $Y(t)$ having these solutions as it's columns:

$$Y(t) = [y_1(t), y_2(t), ..., y_N(t)]$$

Definition.

The determinant

$$w(t) = \det Y(t) = \det\left[y_1(t), y_2(t), ..., y_N(t)\right]$$

is called Wronskian associated with solutions $y_1(t), y_2(t), ..., y_N(t)$.

Proposition 2.7 part (2) - Abel - Liouville's formula

Wronskian $w(t)$ associated with solutions $y_1(t), y_2(t),...,y_N(t)$ to the system $x'(t) = A(t)x(t)$ satisfies the following relations:

$$w(t) = w(\tau)\det\Phi(t,\tau)$$

In points t where $A(t)$ is continuous it satisfies the differntial equation

$$w'(t) = tr(A(t))w(t)$$

and therefore

$$w(t) = w(\tau)\exp\left(\int_{\tau}^{t} tr(A(s)ds\right) \tag{16}$$

for all $t \in J$.

Proof.

We use here that $y_k(t) = \Phi(t,\tau)y_k(\tau)$ and therefore $Y(t) = \Phi(t,\tau)Y(\tau)$. It implies that

$$w(t) = \det Y(\tau) \det \Phi(t,\tau) = w(\tau) \det \Phi(t,\tau)$$

giving the first statement of the Proposition.

We denote by $\varphi_k(t)$ columns in $\Phi(t,\tau)$, so that $\Phi(t,\tau) = [\varphi_1(t), \varphi_2(t), ..., \varphi_N(t)]$.Then we apply the Lemma **about the derivative of a determinant of a matrix valued function** to the case $B(t) = \Phi(t,\tau)$. A direct substitution implies that

$$\frac{\partial}{\partial t}\left(\det \Phi(t,\tau)\right) = \sum_{k=1}^{N} \det\left(U_k(\Phi(t,\tau))\right) = \sum_{k=1}^{N} \det\left(\left[\varphi_1(t), ..., \frac{\partial}{\partial t}\left(\varphi_k(t)\right), ..., \varphi_N(t)\right]\right)$$

where the k-th column in $U_k(\Phi(t,\tau))$ is $\frac{\partial}{\partial t}\left(\varphi_k(t)\right)$ and other columns are columns $\varphi_j(t)$, $j \neq k$, $j = 1, ...N$ from $\Phi(t,\tau)$.

$\frac{\partial}{\partial t}\left(\varphi_k(t)\right) = A(t)\varphi_k(t)$, because $\varphi_k(t)$ are solutions to the system $x'(t) = A(t)x(t)$. We assume here that τ is not a point of discontiuity for $A(t)$. It leads to the more explicit expression:

$$\frac{\partial}{\partial t}\left(\det \Phi(t,\tau)\right) = \sum_{k=1}^{N} \det\left(U_k(\Phi(t,\tau))\right) = \sum_{k=1}^{N} \det\left(\left[\varphi_1(t), ..., A\left(\varphi_k(t)\right), ..., \varphi_N(t)\right]\right)$$

Then $t = \tau$, into the last formula for we arrive to

$$\left.\frac{\partial}{\partial t}\left(\det \Phi(t,\tau)\right)\right|_{t=\tau} = \sum_{k=1}^{N} \det\left(\left[e_1, ..., A(\tau)e_k, ..., e_N\right]\right)$$

because $\Phi(\tau,\tau) = I = [e_1, ..., e_k, ..., e_N]$. Observe that $A(\tau)e_k = A(\tau)_{kk}$ - is the k-th diagonal element in A.Matrices under the determinant sign in the last formula are diagonal with all elements equal to one except one equal to $A(\tau)_{kk}$. Its determinant is the product of diagonal elements $\det\left(\left[e_1, ..., A(\tau)e_k, ..., e_N\right]\right) = A(\tau)_{kk}$.Therefore

$$\left.\frac{\partial}{\partial t}\left(\det \Phi(t,\tau)\right)\right|_{t=\tau} = \sum_{k=1}^{N} \det\left(\left[e_1, ..., A(\tau)_{kk}, ..., e_N\right]\right) = \sum_{k=1}^{N} A_{kk}(\tau) = \mathrm{tr}A(\tau)$$

Therefore

$$w'(\tau) = w(\tau)\mathrm{tr}A(\tau)$$

The argument given here applies to any $\tau \in J$ that is not a point of discontinuity for $A(t)$. The expression

$$w(t) = w(\tau)\exp\left(\int_{\tau}^{t} tr(A(s)ds\right)$$

follows by integration of the differential equation for $w(t)$ using method of integrating factor applied to a scalar first order linear equation.■

Interesting observations and examples with applications of Abel - Liouville's formula.

The geometric meaning of determinant $\det(C)$ of the matrix $C = [c_1, ...c_N]$ with columns $c_1, ...c_N$ is volume of the parallelepiped V build on vectors $c_1, ...c_N$:

$$|\det(C)| = vol\,(V)$$

One can definem V formally as $V = \left\{x \in \mathbb{R}^N : \quad x = \sum_{k=1}^{N} a_k\, c_k, \quad a_k \in [0,1], k = 1, ..., n\right\}$.

It implies that the Abel - Liouvilles formula gives an exact description of how for example the volume of a unique cube build on standard basis vectors $e_1, ..., e_N$ given at the initial time τ is changing by the "flow" described by the transition matrix function $\Phi(t, \tau)$.

Corollary 2.33, p. 59

We consider a periodic linear system $x'(t) = A(t)x(t)$, $A(t + p) = A(t)$.

If $\int_0^p \mathrm{tr}(A(s)ds$ has a positive real part, then the equation has a solution $x(t)$ that is unbounded, or formulating it more formally, the upper limit of it's norm is infinity: $\limsup_{t\to\infty} \|x(t)\| = \infty$

Proof.

Consider a unite cube build on standard base vectors $e_1, ..., e_N$ at time $t = 0$. Consider how the volume Vol(t) of this cube changes under the action of the linear transformation by the transfer matrix $\Phi(t, 0)$ of our periodic system. Point out that $I = [e_1, ..., e_N]$ .It implies that the figure of interest is the parallelepiped build on columns of the transfer matrix $\Phi(t, 0)$.

According to Abel - Liouville's formula and considerations before

$$\begin{aligned} \mathrm{Vol}(t) &= |\det(\Phi(t,0)\, I)| = \left|\det(\Phi(0,0)) \exp\left(\int_\tau^t tr(A(s)ds\right)\right| = \\ \left|\exp\left(\int_\tau^t tr(A(s)ds\right)\right| &= \left|\exp\left(\mathrm{Re}\left(\int_\tau^t tr(A(s)ds\right)\right)\right| \end{aligned}$$

Therefore, if Re $\left(\int_0^p tr(A(s)ds\right) > 0$ then

$$\mathrm{Vol}(p) = |\det(\Phi(p,0)\,)| = \left|\exp\left(\mathrm{Re}\int_0^p tr(A(s)ds\right)\right| > 1.$$

On the other hand $\det(\Phi(p, 0))$ is a product of eigenvalues μ_k to the monodromy matrix $\Phi(p, 0)$ with multiplicities m_k :

$$|\det(\Phi(p,0)| = \prod_{k=1}^{s} |\mu_k|^{m_k} > 1$$

To have this product greater than 1 we must have at least one eigenvalue μ_p with $\left|\mu_p\right|$ >1. Therefore, according to one of Floquet theorems, there is a solution $x(t)$ that is not bounded and therefore $\limsup_{t\to\infty} \|x(t)\| = \infty$. For example we can choose the initial condition $x(0) = v_p$ with v_p being the eigenvector corresponding to the eigenvalue μ_p.Then the solution $x(t)$ is be unbounded and $\limsup_{t\to\infty} \|x(t)\| = \infty$.

We give also a geometric interpretation of this result. One of the main properties of periodic system is that $\Phi(np, 0) = [\Phi(p, 0)]^n$. Therefore

$$\mathrm{Vol}(np) = |\det([\Phi(p,0)]^n\,)| = |\det([\Phi(p,0)]\,)|^n = \left[\exp\left(\mathrm{Re}\left(\int_0^p tr(A(s)ds\right)\right)\right]^n$$

If $\mathrm{Re}\left(\int_0^p tr(A(s)ds\right) > 0$ then $\exp\left(\mathrm{Re}\left(\int_0^p tr(A(s)ds\right)\right) > 1$. It implies that

$$\lim_{n\to\infty} \mathrm{Vol}(n\,p) = \infty$$

Therefore along the sequence of times$\{t = np, \quad n = 1, 2, 3, ...\}$ $\mathrm{Vol}(n\,p)$ is unbounded. It implies also that

$$\limsup_{t\to\infty} \|\mathrm{Vol}(t)\| = \infty$$

The fact that $\lim_{n\to\infty} \mathrm{Vol}(n\,p) = \infty$ implies that the diameter $D(n\,p)$ of the parallelepiped build on columns of $\Phi(n\,p, 0)$ calculated at these discrete time points, also tends to infinity: $\lim_{n\to\infty} D(n\,p) = \infty$. It in turn means that there should be a solution that has the property $\limsup_{t\to\infty} \|x(t)\| = \infty$.