1 Lyapunov stability theory §5.1

The pioneering work by Lyapunov on stability theory where both the idea of linearization and the idea of test functions were introduced and developed, was his Ph.D thesis published in 1892 and translated to French in 1907.

Consider an autonomous system $x' = f(x)$ with $f : G \to \mathbb{R}^N$, $G \subset \mathbb{R}^N$ open. We suppose that f is a locally Lipschitz continuous function, so the existence and uniqueness of maximal solutions to I.V.P. are valid.

We repeat for convenience definitions of stable and unstable equilibrium points (considered here at the origin to make it simpler to apply the construction with Lyapunov functions)

Definition

An equilibrium point $0 \in G$ of the system is said to be stable if for each $\varepsilon > 0$, there is $\delta > 0$ such that for any ξ taken in the ball $B(\delta, 0) = \{\xi \in \mathbb{R}^N, \quad |\xi| < \delta\}$ the maximal solution $x(t) = \varphi(t, \xi) : I_\xi \to G$ on the maximal interval I_ξ with initial data $x(0) = \xi$ and $0 \in I_\xi$ will stay in the ball $B(\varepsilon, 0)$: $\|\varphi(t, \xi)\| < \varepsilon$ for all $t \in I_\xi \cap \mathbb{R}^+$. In fact $\mathbb{R}^+ \subset I_\xi$ in this case.

Definition

The function $V : U \to \mathbb{R}$, U - open, containing the origin $0 \in U$, is said to be positive definite in U, if $V(0) = 0$ and $V(z) > 0$ for $\forall z \in U, z \neq 0$.

Lyapunov's theorem on stability

Theorem. Th.5.2, p.170

Let 0 be an equilibrium point for the system above and there is a positive definite continuously differentiable, $C^1(U)$ function $V : U \to \mathbb{R}$, such that $U \subset G$, $0 \in U$ and $V_f(z) = \nabla V \cdot f(z) \leq 0 \; \forall z \in U$, then 0 is a stable equilibrium point.

Remark.

A function V with these propertie is usually called the Lyapunov function of the system.

Proof.

Take an arbitrary $\varepsilon > 0$ such that $B(\varepsilon, 0) \subset U$. Let $\alpha = \min_{z \in S(\varepsilon, 0)} V(z)$ be a minimum of the continuous function V on the boundary of $B(\varepsilon, 0)$, that is the sphere $S(\varepsilon, 0) = \{z : |z| = \varepsilon\}$ and is a compact set (closed and bounded). Then $\alpha > 0$ because $V(z) > 0$ outside the equilibrium point 0.

By continuity of the function V and the fact that $V(0) = 0$ one can find a $0 < \delta < \varepsilon$ such that $\forall z \in B(\delta, 0)$ we have $V(z) < \alpha/2$.

On the other hand for any part of the trajectory $x(t) = \varphi(t, \xi)$, inside U the function $V(\varphi(t, \xi))$ is *non-increasing* because $\frac{d}{dt} V(\varphi(t, \xi)) = (\nabla V \cdot f)(x(t)) \leq 0$. Therefore all trajectories $\varphi(t, \xi)$ with initial points $\xi \in B(\delta, 0)$ satisfy $V(\xi) < \alpha/2$. Therefore $V(\varphi(t, \xi)) < \alpha/2$ and cannot reach the sphere $S(\varepsilon, 0)$ where $V(z) \geq \alpha = \min_{z \in S(\varepsilon, 0)} V(z)$. Therefore any such trajectory stays within the ball $B(\varepsilon, 0)$ and by the definition, the origin 0 is stable. It implies also that $\mathbb{R}^+ \subset I_\xi$, where I_ξ is the maximal interval for initial point ξ, because the trajectory stays inside a compact set. ∎

Remark. The definition of stability and proofs of the theorems are exactly the same if we take an arbitrary equilibrium point x_0 instead of the origin and use balls $B(\varepsilon, x_0)$ around x_0.

Example.

Investigate stability of the equilibrium point in the origin for the following system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - x_2^3 \end{aligned}$$

that follows from the second order equation $x'' + (x')^3 + x = 0$.

Try the simple test function $V(x_1, x_2) = x_1^2 + x_2^2$. It is positive definite.

We check the sign of the derivative of V along trajectories of solutions: $V_f(x_1, x_2) = (\nabla V \cdot f)(x_1, x_2) = 2x_1 x_2 + 2x_2 (-x_1 - x_2^3) = -4x_2^4 \leq 0$.

Point out that $V_f(x_1, x_2) = 0$ along the x_1 axis where $x_2 = 0$, not only in the origin!!!

Example. One dimensional Newton equation.

Consider a similar example $mx'' + g(x) = 0$, $xg(x) > 0$, $x \neq 0$, $g(0) = 0$. Suppose that $\int_0^x g(s)ds \to \infty$ as $t \to \infty$.

It describes a spring with non-linear force $g(x)$. It can be rewritten as a system of ODE's of the first order.

$$\begin{aligned} x_1 &= x \\ x' &= x_1' = x_2, \\ mx_2' &= -g(x_1) \end{aligned}$$

Consider the test function $V(x_1, x_2)$:

$$V(x_1, x_2) = \frac{m}{2}(x_2)^2 + \int_0^{x_1} g(s)ds$$

representing the energy of the system, consisting of two terms: the kinetic energy $\frac{m}{2}(x')^2$ and the potential energy $G(x) = \int_0^x g(s)ds$.

Point out that V is positive definite because of the limitation $xg(x) > 0$, $x \neq 0$.

Consider the derivative V_f of V along trajectories

$$\begin{aligned} (\nabla V \cdot f)(x_1, x_2) &= \left(\frac{\partial}{\partial x_1}V\right) f_1 + \left(\frac{\partial}{\partial x_2}V\right) f_2 \\ &= g(x_1)x_2 + mx_2 \left(-\left(\frac{1}{m}\right)g(x_1)\right) = 0 \end{aligned}$$

The Lyapunov stability theorem implies that the origin is a stable equilibrium point.

We point out also that $(\nabla V \cdot f)(x_1, x_2) = 0$ is zero everywhere.

Definition

Functions that satisfy the relation $(\nabla V \cdot f)(x_1, x_2) = 0$ and are therefore constant on trajectories of the system $x' = f(x)$ are called *first integrals* of the system.

We can express level sets $V(x_1, x_2) = h$ of the first integral V in the example above as

$$x_2 = \pm\sqrt{\frac{2}{m}(h - G(x_1))}$$

that is valid in points where the expression under the root is non-negative.

Proposition. 4.54, p. 161

If the first integral V has level sets that are closed curves that do not contain equilibrium points, these curves are orbits of periodic solutiuons. This idea is almost the only constructive method to calculate periodic orbits for non-linear systems in plane.

Pointing out that $G(x_1) = \int_0^{x_1} g(s)ds$ in the example above is monotone with respect to $|x_1|$, we conclude that those level sets of $V(x_1, x_2)$ that are closed curves and contain no equilibrium points must be orbits of periodic solutions, according to Poincare-Bendixson theorem. It implies in particular that the origin is not asymptotically stable equilibrium point in this example.

Example. Non-linear pendulum without friction.

A particularly interesting example in the form similar to the last one is the equation for pendulum that we considered earlier by using method with linearization.

$$\theta'' = -\frac{g}{l}\sin\theta$$

Let $k^2 = \frac{g}{l}$

$$\begin{aligned}\theta' &= \psi\\ \psi' &= -k^2\sin\theta\end{aligned}$$

The function $V(\theta, \psi)$

$$V(\theta, \psi) = \frac{\psi^2}{2} + G(\theta)$$

with $G(\theta) = k^2(1 - \cos\theta)$ is the first integral of the system describing the pendulum.

Level sets of the function $V(\theta, \psi) = h$ consist of the orbits of the system

$$\psi = \pm\sqrt{2\left(h - G(\theta)\right)}$$

. For $0 < h < 2k^2$ level sets are periodic orbits. For $h > 2k^2$ level sets are wave-looking orbits of trajectories corresponding to the pendulum rotating around the pivot. There are also level sets corresponding to $h = 2k^2$ and consisting of unstable equilibrium points and orbits connecting them and corresponding to trajectories that tend to the upper non-stable equilibrium and not rotating further.

We draw several level sets for the function $\frac{y^2}{2} + 1 - \cos(x) = h$:
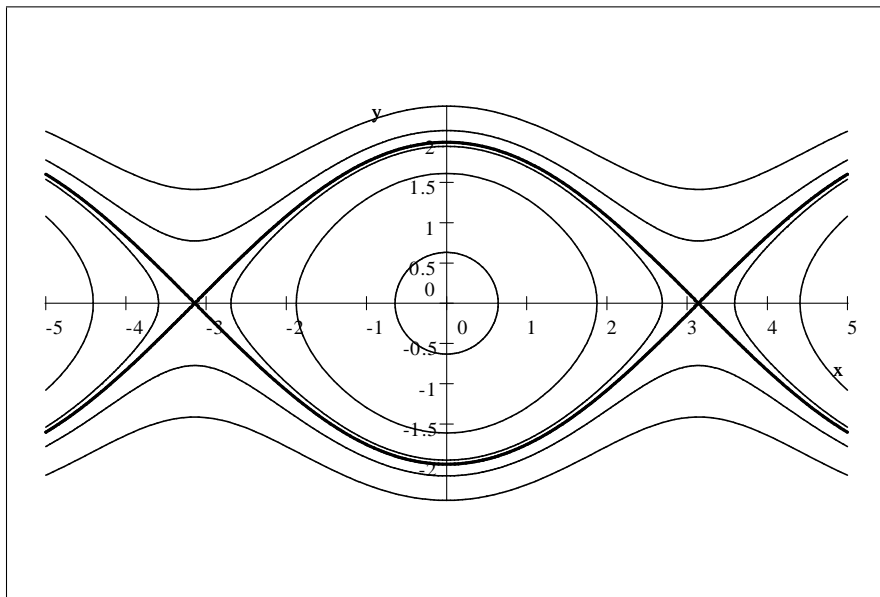
Theorem. Asymptotic stability by Lyapunovs functions. Cor. 5.17, p.185,

Let 0 be an equilibrium point for the system above and let V be a positive definite, continuously differentiable function $V : U \to \mathbb{R}$, such that $U \subset G$, U - open, $0 \in U$, and $V_f(z) = \nabla V \cdot f(z) < 0$ (strict inequality outside the origin!) $\forall z \in U$, $z \neq 0$, then 0 is an asymptotically stable equilibrium point.

Definition. Lyapunov functions satisfying conditions in this theorem are often called *strong Lyapunov functions.*

Proof.

In the course book this theorem is considered as a corollary to a more general LaSalle's invariance principle. We give here an independent proof to asymptotic stability. By the Lyapunov's stability theorem the origin is a stable equilibrium and there is a ball $B(r, 0) \subset U$ such that for any $\xi \in B(r, 0)$, $\mathbb{R}^+ \subset I_\xi$, where I_ξ is the maximal interval for initial point ξ. Therefore we need only to show that the origin is an attractor. Namely we need to show that there is a ball $B(r, 0) \subset U$, such that for any $\xi \in B(r, 0)$ it follows that $\varphi(t, \xi) \to 0$ as $t \to \infty$.

It suffices to show that $\lim_{t \to \infty} V(\varphi(t, \xi) = 0$ because V is continuous and is positive outside the origin, where $V(0) = 0$. It is easy to see by a contradiction argument, because if $\varphi(t, \xi)$ does not tend to the origin, then there is a sequence of times $t_k \to \infty$ as $k \to \infty$ such that $\|\varphi(t_k, \xi)\| > \varepsilon > 0$. It implies that $V(\varphi(t_k, \xi)) > q > 0$ for some positive q. But it is not compatible with suppositionthat $\lim_{t \to \infty} V(\varphi(t, \xi) = 0.\square$

Now we continue proving $\lim_{t \to \infty} V(\varphi(t, \xi) = 0$. By conditions of the theorem $\frac{d}{dt} V(\varphi(t, \xi) < 0$, therefore $0 \leq V(\varphi(t, \xi)$ is a monotone strictly decreasing function of t and must have a limit $\lim_{t \to \infty} V(\varphi(t, \xi) = \eta$ as $t \to \infty$.

Suppose that this limit is not zero: $\eta > 0$. Then $V(\varphi(t, \xi) \geq \eta > 0$ for all $t \geq 0$ because $V(\varphi(t, \xi)$ is monotone.

Now we will find a ball $B(\delta, 0)$, $\delta < r$ around the origin so small that the trajectory $\varphi(t, \xi)$ cannot reach it. The idea is that outside this ball (where our trajectory $\varphi(t, \xi)$ is situated) the decreasing rate for $V(\varphi(t, \xi)$ along the trajectory is never close to zero. This fact would lead us to a contradiction with out supposition.

Continuity of V and the fact that $V(0) = 0$ imply that there is a ball $B(\delta, 0)$, $\delta < r$ such that $0 \leq V(z) < \eta/2$ for all $z \in B(\delta, 0)$. Hence $\varphi(t, \xi)$ cannot reach it: $\|\varphi(t, \xi)\| \geq \delta$ for all $t \geq 0$, because $V(\varphi(t, \xi) \geq \eta > 0$ for all $t \geq 0$ by our supposition that $V(\varphi(t, \xi) \searrow \eta$ as $t \to \infty$.

Now we will estimate the smalles rate of decrease for $V(\varphi(t, \xi)$ that follows from our conclusions. Consider the closed spherical slice $S = \{z : \delta \leq \|z\| \leq r\}$ and point out that $\gamma = \min_{z \in S} (-V_f(z)) > 0$ exists because V_f is continuous.

$\gamma > 0$ by the condition of the theorem that $V_f < 0$ outside the origin. Therefore

$$-\frac{d}{dt} V(\varphi(t, \xi)) \geq \gamma = \min_{z \in S} (-V_f(z))$$

and

$$\frac{d}{dt} V(\varphi(t, \xi)) \leq -\gamma$$

By integration from 0 to t we arrive to

$$V\left(\varphi(t,\xi)\right) - V(\xi) \leq -\gamma t \to -\infty$$

as $t \to \infty$ that contradicts to the supposition that $V(z) \geq 0$.

It implies that our supposition that $\lim_{t \to \infty} V(\varphi(t,\xi)) = \eta > 0$ was wrong and that $\lim_{t \to \infty} V(\varphi(t,\xi)) = 0$. As we pointed out at the beginning of the proof, the last fact implies, that $\lim_{t \to \infty} \varphi(t,\xi) = 0$ and therefore the origin is an attractor and is an asymptotically stable equilibrium point.∎

Remark.

This theorem on asymptotic stability has a (very difficult!) inversion (proven in 1949,1956) by José Luis Massera, Uruguay, stating that for any system with an asymptotically stable equilibrium point, there is a "strong" Lyapunov function V such that $V_f(z) < 0$ in a neighborhood of this equilibrium point (outside the point $z = 0$ itself).

Definition. Region of attraction for an asymptotically stable equilibrium point.

A domain $U \subset G$ is called the attraction region for an asymptotically stable equilibrium point $x_* \in U$ if for any $\xi \in U$, the maximal existence interval I_ξ of the the solution $x(t) = \varphi(t,\xi)$ contains $\mathbb{R}^+ \subset I_\xi$ and $\varphi(t,\xi) \to x_*$ as $t \to \infty$.
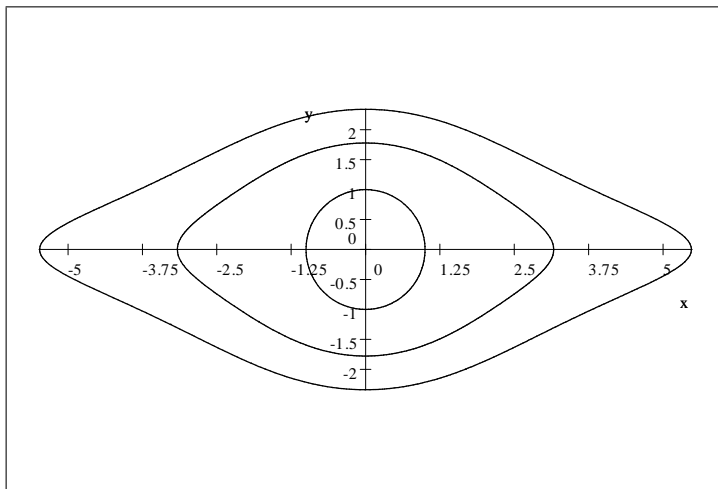
Example. Consider the system of equations

$$
\begin{aligned}
y_1' &= -y_1^3 - 2y_1 y_2^2 \\
y_2' &= y_1^2 y_2 - y_2^3
\end{aligned}
$$

Investigate stability of the equilibrium point in the origin finding a suitable Lyapunov function. Consider the following test function:

$$V(y_1, y_2) = y_1^2 + y_1^2 y_2^2 + y_2^4$$

The test function V is positive definite. We draw several level sets for $V(x,y) = x^2 + x^2 y^2 + y^4 = h$, for $h = 1, 20, 30$.

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We choose the form of the test function in such a way that on the level set curves of this function velocities $f(y_1, y_2)$ point inward: $(\nabla V \cdot f)(y_1, y_2) < 0$. We have chosen the term y_2^4 having $\frac{\partial}{\partial y_2}(y_2^4) = 4y_2^3$ that after multiplication with the term $-y_2^3$ from f_2 gives a "good" negative term $-4y_2^6$. Similarly $\frac{\partial}{\partial y_1}(y_1^2) = 2y_1$ multyplied by the term $-y_1^3$ from f_1 gives a good negative term $-2y_1^4$. The tricky step is to play with "bad" indefinite mixed products in such a way that they (in the best case!) give no terms in $(\nabla V \cdot f)(y_1, y_2)$ with indefinite sign.

$$
\begin{aligned}
V_f(y_1, y_2) &= (\nabla V \cdot f)(y_1, y_2) = \\
&\quad \left(2y_1 + 2y_1 y_2^2\right)\left(-y_1^3 - 2y_1 y_2^2\right) + \\
&\quad + \left(2y_1^2 y_2 + 4y_2^3\right)\left(y_1^2 y_2 - y_2^3\right) \\
&= -2y_1^4 - 4y_1^2 y_2^2 - 2y_1^4 y_2^2 - 4y_1^2 y_2^4 + 2y_1^2 y_2^4 - 4y_2^6 + 2y_1^4 y_2^2 \\
&= -2y_1^4 - 4y_2^6 - 4y_1^2 y_2^2 - 2y_1^2 y_2^4 \\
&= \left(-y_1^4 - 2y_2^6 - 2y_1^2 y_2^2 - y_1^2 y_2^4\right) 2 \\
&\leq \left(-y_1^4 - 2y_2^6\right) 2 < 0, \quad (y_1, y_2) \neq (0, 0)
\end{aligned}
$$

Therefore according to the last theorem, the origin $(0, 0)$ is an asymptotically stable equilibrium point.

Remark

One can arrive to indefinite terms after calculation of V_f. It is still not the end of hope. One can check that these indefinite terms are not large and might be compensated by negative definite terms in the expression for V_f. For example the expression $-x^2 - y^2 + xy < 0$ for $(x, y) \neq (0, 0)$ because $2|xy| \leq x^2 + y^2$. One can also use criteria for positive and negative definite quadratic forms in such problems.

Lyapunov's theorem on instability

We give here a slightly weaker variant of the instability theorem comparing with one in the book. An advantage of the variant here is that it suggests a more constructive proof.

Students are free to choose any of these two variants at the examination.

Definition.

An equilibrium point $0 \in G$ of the system is said to be unstable if it is not stable.

Explicit version of the same definition.

There is a ball $B(R, 0) \subset G$ and for any $\delta > 0$ there is a point $\xi \in B(\delta, 0)$ such that for the trajectory $\varphi(t, \xi)$ starting in ξ there is time $t_* \in I_\xi$ such that $\varphi(t, \xi) \notin B(R, 0)$. Another reformulation of this definition is possible.

Another explicit version of the same definition

There is a ball $B(R, 0) \subset G$ and a sequence of points $\{x_n\}_{n=1}^\infty$ such that $\lim_{n \to \infty} x_n = 0$ such that for each maximal solution $\varphi(t, \xi)$ with initial data $\xi = x_n$ there is time $t_* \in I_\xi$ such that $\varphi(t, \xi) \notin B(R, 0)$.

Theorem. On a criterion of instability of an equilibrium using test functions.

Let the origin 0 - be the equilibrium point of the system $x' = f(x)$. Assume that there is a neighbourhood $U \subset G$, $0 \in U$ and a continuously differentible $C^1(U)$ function $V : U \to \mathbb{R}$ satisfying the following hypotheses.

1) $V_f(z) = \nabla V \cdot f(z) > 0$ for every $z \in U$, $z \neq 0$
2) For every $\delta > 0$ there exists $z \in U$ with $\|z\| < \delta$ and $V(z) > 0$
3) $V(0) = 0$.

Then the origin 0 is an unstable equilibrium.

Remark. The Theorem Th. 5.7, p. 174 formulted in the book is stronger. It has the same conclusion with the condition 2) changed to a weaker one:

2) $V_f(z) = \nabla V \cdot f(z) > 0$ for every point $z \in U$, where $V(z) > 0$, and 3) is not required.

Proof of the weaker variant of the Theorem

The idea of the proof is to show that any trajectory starting from a point ξ arbitrarily close to 0 where $V(\xi) > 0$ will leave a fixed ball $B(R/2, 0)$ such that a larger ball $\overline{B(R, 0)} \subset U$.

We point out that for any part of the trajectory $\varphi(t, \xi)$ of the maximal solution in U the function $V(\varphi(t, \xi))$ is monotone increasing because $\frac{d}{dt} V(\varphi(t, \xi)) = V_f(\varphi(t, \xi)) > 0$.

It means that $\varphi(t, \xi)$ stays outside the origin because $V_f(z)$ is continuous and $V_f(0) = 0$. It in turn means that $(\nabla V \cdot f)(\varphi(t, \xi)) = \frac{d}{dt} V(\varphi(t, \xi) \geq K > 0$ for all $t \in I_\xi \cap \mathbb{R}^+$.

To prove this inequality one can carry out a more formal argument that follows.

Let $\xi \in B(R, 0)$ be an arbitrary point where $V(\xi) > 0$. V is a continuous function and $V(0) = 0$. It implies that there is $0 < \varepsilon < R/2$ such that $V(z) < V(\xi)/2$ for $\|z\| < \varepsilon$.

Therefore the trajectory $\varphi(t, \xi)$ must stay outside the ball $B(\varepsilon, 0)$ for all $t \in I_\xi \cap \mathbb{R}^+$.

The function $(\nabla V \cdot f)(z)$ is continuous in U and must attaint its minumum $K = \min_{z \in \overline{B(R,0)} \backslash B(\varepsilon, 0)} (\nabla V \cdot f)(z)$ on the compact set $\overline{B(R, 0)} \backslash B(\varepsilon, 0)$ that is a slab between two spheres. The number K is positive $K > 0$ because $(\nabla V \cdot f)(z) > 0$ for $z \in U$ outside the origin.

Therefore
$$(\nabla V \cdot f)(\varphi(t,\xi)) = \frac{d}{dt}V(\varphi(t,\xi)) \geq K > 0, \quad \forall t \in I_\xi \cap \mathbb{R}^+$$

and by the integration of the left and right hand side over $[0,t]$ we get

$$V(\varphi(t,\xi)) \geq Kt + V(\xi), \quad \forall t \in I_\xi \cap \mathbb{R}^+$$

There are two possibilities depending on if $I_\xi \cap \mathbb{R}^+$ is a bounded interval or $\mathbb{R}^+ \subset I_\xi$. In the first case the trajectory $\varphi(t,\xi)$ must leave any compact in G in particlular the ball $\overline{B(R,0)}$. In the second case having possibility to take t arbitrary large in the inequality $V(\varphi(t,\xi)) \geq Kt + V(\xi)$ leads to conclusion, that for some time $t_* > 0$ large enough $V(\varphi(t_*,\xi))$ will become larger than $\max_{z \in \overline{B(R/2,0)}} V(z)$ - the maximum of $V(z)$ over the half ball $\overline{B(R/2,0)}$. It means that the point $\varphi(t_*,\xi)$ of the trajectory must be outside the ball $\overline{B(R/2,0)}$ at such time t_*.

Therefore according to the definition, the origin 0 is an unstable equilibrium, because there are trajectories starting arbitrarily close to the equilibrium 0, such that they are move outside the ball $\overline{B(R/2,0)} \subset G$ at some time t_*. ∎

Remark. If we suppose in the formulation of the theorem above that $V(z) > 0$ for all $z \in U$, $z \neq 0$, then the origin is a repeller, meaning that for some ball $B(R,0)$ around the origin, any solution $x(t) = \varphi(t,\xi)$ with $\xi \in B(R,0)$ will leave this ball in finite time.