MATEMATIK	Datum: 2018-01-03	Tid: 8.30-12:30
GU, Chalmers	Hjälpmedel: - Inga	
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Tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate and give a proof to the theorem about the dimension of the space of solutions to a linear system of ODEs. **(4p)**

   Check

2. Formulate and give a proof to the theorem on stability of equilibrium points of autonomous non-linear ODEs by linearization with Hurwitz variational matrix. **(4p)**

3. Consider the following initial value problem: $y' = \sin(y)t^2$; $y(1) = 2$.

   a) Reduce the initial value problem to an integral equation and give a general description of iterations approximating the solution as in the proof to the existence and uniqueness theorem by Picard and Lindelöf. **(2p)**

   b) Find a time interval such that these approximations converge to the solution of the initial value problem. **(2p)**

   Solution.

   We introduce an integral equation equivalent to the ODE $y' = f(t, y)$ by the integration of the right and left hand sides in the equation:

$$y(t) = y(1) + \int_1^t f(s, y(s))\mathrm{d}s.$$

   Taking $y_0(t) = y(1)$ we define Picard iterations by the recurrense relation

$$y_{n+1}(t) = y(1) + \int_1^t f(s, y_n(s))\mathrm{d}s.$$

   For the particular equation it looks as

$$y_{n+1}(t) = y(1) + \int_1^t \sin(y_n(s))s^2\mathrm{d}s = \mathbb{K}(y_n, t).$$

   One proves the existence and uniqueness theorem by showing that at some time interval the integral operator $\mathbb{K}(y, t) = y(1) + \int_1^t \sin(y(s))s^2\mathrm{d}s$ in the right hand side is a contraction:

$$\sup_{t\in[1,T]} |\mathbb{K}(w, t) - \mathbb{K}(u, t)| < \alpha \sup_{t\in[1,T]} |w(t) - u(t)|$$

   $\alpha < 1$, in a ball $\sup_{t\in[1,T]} |w(t) - y(1)| \leq R$ in the space of continuous functions, and maps this ball into itself:

$$\sup_{t\in[1,T]} |\mathbb{K}(w, t) - y(1)| \leq R$$

   and applying the Banach contraction theorem to $\mathbb{K}(y, t)$.

   We estimate first $\sup_{t\in[1,T]} |\mathbb{K}(w, t) - \mathbb{K}(u, t)|$ for continuous functions u and w such that $\sup_{t\in[1,T]} |w(t) - y(1)| \leq R$ and

$\sup_{t\in[1,T]} |u(t) - y(1)| \leq R$. Point out that $\sup_{t\in[1,T]} |w(t)| \leq y(1) + R$. We will find T such that the contraction property is valid:

$$\sup_{t\in[1,T]} \left| \int_1^t \sin(w(s))s^2 \mathrm{ds} - \int_1^t \sin(u(s))s^2 \mathrm{ds} \right| \leq \alpha \sup_{t\in[1,T]} |w(t) - u(t)|, \quad \alpha < 1$$

We carry out elementary estimates using the triangle inequality and intermediate value theorem for sin. $\left| \int_1^t \sin(w(s))s^2 \mathrm{ds} - \int_1^t \sin(u(s))s^2 \mathrm{ds} \right| = \int_1^t |(\sin(w(s)) - \sin(u(s))| \, s^2 ds =$

$\int_1^t |(w(s) - u(s)) \cos(\theta(s))| \, s^2 ds \leq (T-1)\, T^2 \sup_{t\in[1,T]} |w(s) - u(s)|$

The argument $\theta(s)$ above is a number between $w(s)$ and $u(s)$ that exists according the intermediate value theorem. It was also used above that $|\cos(\theta)| \leq 1$. Therefore to have the contraction property we need to have $(T-1)\, T^2 < 1$.

For a function w with $\sup_{t\in[1,T]} |w(t) - y(1)| \leq R$ we need that $|\mathbb{K}(w,t) - y(1)| \leq R$

The following estimate leads to another bound for T: $\sup_{t\in[1,T]} |\mathbb{K}(w,t) - y(1)| \leq \sup_{t\in[1,T]} \left| \int_1^t \sin(w(s))s^2 ds \right| \leq$ $(T-1)\, T^2 \leq R$.

Therefore the time interval must satisfy estimates $(T-1)\, T^2 < 1$ and $(T-1)\, T^2 < R$ to have convergence of Picard iterations in the ball $\sup_{t\in[1,T]} |w(t) - y(0)| \leq R$. Taking $R = 1$ we get an optimal estimate $(T-1)\, T^2 < 1$ that is satisfied for example for $T = 1.4$:

$\alpha = 0.4(1.4)(1.4) = 0.784$

4. Consider the following system of ODE: $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$, with a constant matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$. Give general solution to the ODE. Find all initial data such that corresponding solutions to the system are bounded. **(4p)**

Solution.

The solution to the initial value problem with arbitrary initial data $\overrightarrow{r}(0)$ is $\overrightarrow{r}(t) = \exp(tA)\overrightarrow{r}(0)$.

The matrix A xas a block diagonal structure $A = \begin{bmatrix} J & \mathbb{O} \\ \mathbb{O} & Z \end{bmatrix}$ where $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a Jordan bloc and $Z = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

Therefore $\exp(tA) = \begin{bmatrix} \exp(tJ) & \mathbb{O} \\ \mathbb{O} & \exp(tZ) \end{bmatrix}$.

$\exp((tJ) = I + tJ + \frac{1}{2}t^2 J^2 + ... = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$,because $J^2 = 0$;

$\exp(tZ) = \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix}$. The last relation can be approved in the following way. Multiplication and addition of matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ satisfies the same rools as multiplication and addition of complex numbers $z = a + ib$. Therefore the matrix $\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$ corresponds through this correspondence to purely imaginaty numbers, and the relation $\exp(ib) = \cos(b) + i\sin(b)$ can be applied leading to the formula for $\exp(tZ)$ above.

General solution to the system of ODEs with initial data $\begin{bmatrix} r_1 & r_2 & r_3 & r_4 \end{bmatrix}^T$ is

$$\overrightarrow{r}(t) = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(2t) & -\sin(2t) \\ 0 & 0 & \sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

It is easy to see that $\overrightarrow{r}(t)$ is bounded if and only if $r_2 = 0$

One can also construct general solution as a linear combination of eigenvectors and generalized eigenvectors:

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ - the eigenvector corresponding to $\lambda = 0$. $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ - the generalized eigenvector corresponding to $\lambda = 0$, $u_3 = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$ the complex eigenvector corresponding to $\lambda = 2i$, and $u_4 = \begin{bmatrix} 0 \\ 0 \\ -i \\ 1 \end{bmatrix}$ the complex eigenvector corresponding to $\lambda = -2i$.

The general solution has the form:

$$\overrightarrow{r}(t) = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_3 \operatorname{Re}\left(\exp(2it) \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} \right) + C_4 \operatorname{Im}\left(\exp(2it) \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} \right) =$$

$$C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_3 \left(\cos(2t) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \sin(2t) \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right)$$

$$+ C_4 \left(\cos(2t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \sin(2t) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

We observe here that the solution has the same as from the formula for $\exp(At)[C_1, C_2, C_3, C_4]^T$ and again that solutions are bounded if and only if $C_2 = 0$.

5. Consider the following system of ODEs: $\begin{cases} x' = & -x + 5y^3 \\ y' = & -x^3 - 3y \end{cases}$.

   Show asymptotic stability of the equilibrium point in the origin and find it's region of attraction.

(4p)

Solution.

Choose a test function $V(x, y) = \frac{1}{4}\left(x^4 + 5y^4\right)$. $V(x)$ is positive definite and

$$\begin{aligned} \nabla V \cdot \overrightarrow{f} &= \nabla \left(\frac{1}{4}\left(x^4 + 5y^4\right) \right) \cdot \begin{bmatrix} -x + 5y^3 \\ -x^3 - 3y \end{bmatrix} = \begin{bmatrix} x^3 \\ 5y^3 \end{bmatrix} \cdot \begin{bmatrix} -x + 5y^3 \\ -x^3 - 3y \end{bmatrix} \\ &= 5x^3y^3 - 15y^4 - x^4 - 5y^3x^3 = -x^4 - 15y^4 \leq 0 \end{aligned}$$

$\nabla V \cdot \overrightarrow{f}(x,y) = 0$ only for $(x,y) = (0,0)$. Therefore the origin is asymptotically stable.

Any region $\{(x,y) : V(x,y) \leq R\}$ with a $R > 0$ is a region of attraction. Pointing out that $V(x,y) \to \infty$ with $\|(x,y)\| \to \infty$ we conclude that the origin is globally asymptoticaly stable and the whole $\mathbb{R}^2$ is the region of attraction for the origin.

6. Show that the system

$$\begin{cases} x' = & y \\ y' = & -x + y\left(1 - x^2 - 2y^2\right) \end{cases}$$

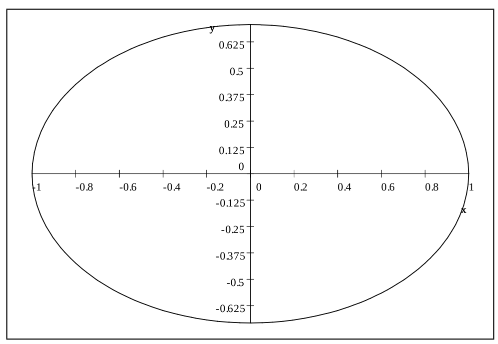
has periodic solutions. **(4p)**

Solution.

Consider a test function $V(x,y) = (x^2 + y^2)$

$\nabla V \cdot f = 2y^2\left(1 - x^2 - 2y^2\right)$

The sign of the derivative of V along trajectories of the system depends on the sign of the expression $\left(1 - x^2 - 2y^2\right)$. Analysing it we observe that trajectories through the points (x,y) outside the ellipse $x^2 + 2y^2 < 1$:



do not leave discs bounded by level sets of $V(x,y) = x^2 + y^2 = const.$ The smallest circle outside this ellipse is $x^2 + y^2 = 1$.

Similarly for points inside this ellipse, trajectories do not enter discs bounded by level sets of $V(x,y)$ (circles). The largest circle inside this ellipse is $x^2 + y^2 = 1/2$.

It implies that the annulus $1/2 < x^2 + y^2 < 1$ is a positively invariant set for this system. It includes no stationary points, because stationary points must have $y = 0$ by the first equation, and in this case $x' \neq 0$ outside the origin. Therefore this annulus must include at least one periodic orbit.

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.32 Assignments + 0.68 Exam$ - the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for the home assignments, and for the total points for the course.