Lecture notes on non-linear ODEs: existence, extension, limit sets, periodic solutions.

Plan

1. Peano theorem on existence of solutions (without proof), Theorem. 4.2, p. 102.

2. Existence and uniquness theorem by Picard and Lindelöf . Th. 4.17, p. 118 (for continuous $f(t,x)$, locally Lipschitz in x),

Th.4.22, p.122 (for piecewise continuous $f(t,x)$, locally Lipschitz in x). (Proof comes in the last week of the course)

3. Maximal solutions. Openess of the maximal existence interval. Prop. 4.4., p. 107.

4. Existence of Maximal solutions. Theorem 4.8.

5. Extensibility of bounded solutions to the boundary time point of the interval. Lemma 4.9, p. 110.

6. Corollary 4.10, p. 111, on solutions eclosed in a compact, implying "infinite" maximal interval.

7. Properties of limits of maximal solutions. Theorem 4.11, p. 112 on the property of solutions with "finite" maximal interval $I_{\max}$, to escape any compact subset C in the space domain G.

8. On infinite existence interval for systems with linear growth estimate for the right hand side. Proposition 4.12, p. 114.

9. Transition map. Definition p. 126. Transition property of the transition map. Translation property for autonomous systems.

Theorem 4.26, p. 126. (similar to Chapman - Kolmogorov relations for transition matrix)

10. Openness of the domain and smoothness of transition map.Theorem 4.29, p. 129. (only idea of the proof is discussed)

11. Autonomous systems. Flows and continuous dependence. §4.6.1. Example 4.33., p. 139.

12. Semi- orbits. Limit sets. p. 142. Positively (negatively) invariant sets p. 142.

13. Properties of ω - limit sets. Theorem 4.38, p. 143

14. Existence of an equilibrium point in a compact positively invariant set. Theorem 4.45, p. 150.

15. Planar systems. Periodic orbits. Poincare-Bendixson theorem. (only idea of the proof is discussed) Theorem 4.46, p. 151.

16. Bendixson's criterion on non existence of periodic solutions.(after lecture notes)

17. First integrals and periodic orbits. Limit cycles. §4.7.2

0.1 Non-linear systems. Existence and uniqueness of solutions.

Second half of the course deals with initial value problems for non-linear systems of ODE's, non-autonomous:

$$x'(t) = f(t, x), \quad f : J \times G \to \mathbb{R}^n; \quad x(\tau) = \xi \tag{1}$$

with $J \subset \mathbb{R}$ - an interval, $G \subset \mathbb{R}^n$, open, $\tau \in J$, $\xi \in G$, f- continuous in $J \times G$, and autonomous systems of ODE's:

$$x'(t) = f(x), \quad f : G \to \mathbb{R}^n; \quad x(\tau) = \xi \tag{2}$$

that are a particular case of (1) with $G \subset \mathbb{R}^n$, open, $\tau \in J = \mathbb{R}$, $\xi \in G$, f- continuous in G, where the right hand side f in the equation is independent of the time variable t running over the whole $\mathbb{R}$. The practical meaning of this kind of systems is that the "velocity" f of the system depends only on the position x, but not on time t. So independently of the starting time τ the output $x(t)$ of an evolution depends only on the shift in time $t - \tau$. It lets to choose always $\tau = 0$ for autonomous systems.

In many situations the equivalent integral form of I.V.P. is convenient to use:

$$x'(t) = \xi + \int_\tau^t f(s, x(s))ds \tag{3}$$

Another option of requirements to f that is considered in the book by Logemann Ryan is that f is supposed to be piecewise continuous in t and locally Lipschitz with respect to x. We will not consider this case systematically in this part of the course.

The fundamental question of existence of solutions is answered by the following Peano theorem (with possibility of non-uniqueness of solutions)

Theorem 4.2, p. 102. Peano theorem.

For each (τ, ξ) in $J \times G$ there exists a solution to (1) defined on a (possibly small) time interval $I \subset J$, $\tau \in I$.

This result implies also the solvability of the problem (2) that is just a particular case.

The proof of this theorem is based on the compactness principle, one of two main approaches in analysis to the existence of solutions to non-linear equations. We will not give a proof here, but will sketch main ideas behind it.

i) One of characteristic properties of compact sets in complete normed spaces is, that any sequence of points $\{z_n\}_{n=1}^{\infty}$ from a compact set C always has a converging subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ with a limit $\lim_{k\to\infty} z_{n_k} = z_*$ that belongs to C: $z_* \in C$.

ii) One approximates solutions to (1) by the explicit Euler method and considers a sequence $\{y_n(t)\}_{n=1}^{\infty}$ of approximations ith the step of finite differences tending to zero with $n \to \infty$.

iii) Considering these approximations on a time interval I including τ and choosing this interval small enough (depending on the absolute value of f around (τ, ξ)), one can show that the approximathions $\{y_n(t)\}_{n=1}^{\infty}$, are uniformly bounded and uniformly continuous on I.

iv) Then basing on the property i) and on iii), one can choose a subsequence $\{y_{n_k}(t)\}_{k=1}^{\infty}$ converging uniformly on I, to a function $y(t)$ in the space of continuous vector valued functions on I , that is a solution to (3) and therefore to (1).$\square$

Exercise. Show that the I.V.P. $x' = \sqrt[3]{x}$; $x(0) = 0$, has non-unique solutions.

The uniqueness of solutions to I.V.P. needs additional requirements on regularity of $f(t, x)$ with respect to x variable. The standard requirement is that $f(t, x)$ is supposed to be locally Lipschitz with respect to the space x variable.

We repeat here the definition of locally Lipschitz functions.

Definition.(p. 115)

Let $D \subset \mathbb{R}^Q$ be a non-empty set A function $g : D \to \mathbb{R}^M$ is said to be locally Lipschitz if for any $z \in D$ there is a set $U \subset D$, realtively open in D, $z \in U$, and a number $L \geq 0$ (which may depend on U) such that

$$\|g(u) - g(w)\| \leq L \|u - w\|, \quad \forall u, w \in U$$

If L is independent of the choice of U, the function is called globally Lipschitz.

Similarly one defines functions locally Lipschitz with respect to a part of variables.

Definition.(p. 118)

Let $G \subset \mathbb{R}^n$ be a non-empty open set, J be an interval in $\mathbb{R}$. A function $f : J \times G \to \mathbb{R}^n$

is said to be locally Lipschitz with respect to $x \in G$ if for any $(\tau, x) \in J \times G$ there is a set $S \times U \subset J \times G$, realtively open in $J \times G$ and a number $L \geq 0$ (which may depend on $S \times U$) such that

$$\|g(s,x) - g(\sigma, y)\| \leq L \|x - y\|, \quad \forall (s,x), (\sigma, y) \in J \times G$$

A theorem that gives conditions for both existence and uniquness of solutions to (1) is called the Picard-Lindelöf theorem

We will prove it in the last week of the course by applying the Banach contraction principle, that is the second main approach in analysis to existence of solutions to non-linear equations.

Theorem. Picard-Lindelöf. Theorem 4.17, p. 118 (variant with continuous f). **Theorem 4.22, p. 122 (variant with piecewise continuous** f).

Let with $J \subset \mathbb{R}$ - an interval, $G \subset \mathbb{R}^n$, open, $\tau \in J$, $\xi \in G$, f be continuous in $J \times G$. If f is locally Lipschitz with respect to its second argument $x \in G$, then there is a unique maximal solution $x : I_x \to \mathbb{R}^n$ to the I.V.P. problem (1). Any other maximal solution with the same initial conditions must coinside with $x(t)$.

Definition. By maximal solution we mean here the solution that cannot be extended to a larger time interval.

A simpler version of this theorem states just that a "local" solution to (1) on a possibly small time interval $I \subset J, \tau \in I$, exists and is unique in the sence that any two solutions x and y must coinside on the intersection of the time intervals I_x and I_y where they are defined.

Proof of local uniqueness uses the integral form of the problem and the argument with Grönvall inequality that was applied two times earlier for lineary systems.

The same argument is used for proving well posedness of the I.V.P., namely that solutions to initial value problem (1) considered as functions of three variables t, τ, ξ: $x(t) = \varphi(t, \tau, \xi)$ are continuous and in fact even locally Lipschitz with respect to all three variables t, τ, ξ.

0.2 Extensions, maximal solutions and their properties.

We consider in this section the problem (1) with f continuous and satisfying conditions in the Peano theorem implying existence (but not uniqueness) of "local solutions $x : I \to \mathbb{R}^n$ on an interval $I \subset J$.

Definition. p. 106.

An extension (proper extension) of the solution x is a solution $\widetilde{x} : \widetilde{I} \to \mathbb{R}^n$ to (1) such that $\widetilde{x}(t) = x(t)$ $\forall t \in I$, $I \subset \widetilde{I}$, $\widetilde{I} \neq I$.

Definition. p. 106. Maximal solution and maximal interval of existence.

The interval I is a maximal interval of existence and x is called maximal solution if x does not have an extension to a larger interval that is a solution to (1).

Plan

• We are going to prove first an important property of maximal intervals (namely that they are relatively open).

• Then we prove an existence theorem for maximal solutions, namely the fact that any solution to (1) can be extended to a maximal solution (an extension with maximal interval of existence). Theorem 4.8, p. 108.

• Then we consider conditions implying that the maximal interval of existence $I_{\max}$ is infinite (if J is $\mathbb{R}$), or "infinite with respect to J" meaning that the maximal solution exists on the whole part of J to the right or to the left of the initial time τ: on $[\tau, \infty) \cap J$ or on $(-\infty, \tau] \cap J$. Corollary 4.10, p. 111.

This important property is based on a technical Lemma 4.9. p. 110 that shows that a solution x defined on an open interval I and having bounded orbit towards future: $O_+ = \{x(t) : x \in [\tau, \sup I)\}$ with closure in G can be extended up to the boundary point and to the closed interval $[\tau, \sup I]$. Similar result is valid for the extension to the boundary point in the "past".

• After that we consider situation opposite to the previous one and describe the behaviour of maximal solutions that have bounded maximal interval $I_{\max}$ (if J is $\mathbb{R}$), or in the case when J is bounded, a maximal interval "bounded with respect to J", $I_{\max}$, not reaching boundaries of J, meaning that $\sup I_{\max} < \sup J$ or $\inf J < \inf I_{\max}$. Theorem 4.11, p.112.

• Then for an equation defined on the whole $\mathbb{R}^n$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ we prove that if f grows not faster than linerarly: $\|f(t,z)\| \leq L(1+\|z\|)$ for t on any compact time interval K, then solutions to (1) exist on $[\tau,\infty) \cap J$ or on $[\tau,\infty)$ if $J = \mathbb{R}$. Proposition 4.12. This result is particularly useful because the condition is easy to check.

The condition in the Proposition 4.12 is not necessary, but simple examples show solutions that blow up in finite time in future or in the past if this condition is not satisfied, as for example the equation $x' = x^2$.

We suggest first simple examples of maximal solutions and maximal intervals that can be calculated explicitely.

Exercise 4.6

$J = [-1,1]$; $G = R$; $\quad f : J \times G \to R.$

$(\tau,\xi) = (0,1)$

$$f(t,z) = \frac{3z^2\sqrt{1-|t|}}{2}$$

$t \in [0,1]$

$$\begin{aligned}
\frac{dz}{dt} &= \frac{3z^2\sqrt{1-t}}{2} \\
\frac{dz}{z^2} &= \frac{3\sqrt{1-t}}{2}dt \\
\frac{-1}{z} &= -(1-t)^{3/2} + C \\
-1 &= -1 + C; \quad (\tau,\xi) = (0,1) \\
C &= 0 \\
z &= \frac{1}{(1-t)^{3/2}}; \quad t \in [0,1)
\end{aligned}$$

$t \in [-1,0]$;

$$\frac{dz}{dt} = \frac{3z^2\sqrt{1+t}}{2}$$

$$
\begin{aligned}
\frac{dz}{z^2} &= \frac{3\sqrt{1+t}}{2}dt \\
\frac{-1}{z} &= (1+t)^{3/2} + C \\
-1 &= 1 + C; \quad (\tau, \xi) = (0, 1) \\
C &= -2 \\
\frac{-1}{z} &= (1+t)^{3/2} - 2 \\
z &= \frac{1}{2 - (1+t)^{3/2}}; \quad t \in [-1, 0];
\end{aligned}
$$

The maximal interval $I_{\max} = [-1, 1)$ - is relatively open in $[-1, 1]$

Exercise 4.7

$J = (-\infty, 1)$; $G = (-\infty, 1)$.

$$f(t,z) = \frac{1}{\sqrt{(1-t)(1-z)}}$$

$$\begin{aligned}
\frac{dz}{dt} &= \frac{1}{\sqrt{(1-t)(1-z)}} \\
\int \sqrt{1-z}dz &= \int \frac{dt}{\sqrt{(1-t)}} \\
\frac{2}{3}(z-1)\left(\sqrt{1-z}\right) &= -2\sqrt{1-t}+C \\
\frac{2}{3}(-1)(1) &= -2+C; \quad t=0,\ z=0 \\
4/3 &= 2-2/3=C \\
\frac{2}{3}(z-1)\left(\sqrt{1-z}\right) &= -2\sqrt{1-t}+\frac{4}{3} \\
\frac{2}{3}(1-z)\left(\sqrt{1-z}\right) &= 2\sqrt{1-t}-\frac{4}{3} \\
(1-z)\left(\sqrt{1-z}\right) &= 3\sqrt{1-t}-2 \\
(1-z)^{3/2} &= 3\sqrt{1-t}-2 \\
(1-z) &= \left(3\sqrt{1-t}-2\right)^{3/2} \\
z &= 1-\left(3\sqrt{1-t}-2\right)^{3/2}
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to 5/9} x(t) &= 1 \\
I_{\max} &= (-\infty, 5/9)
\end{aligned}$$

$I_{\max}$ is open.

Theorems 4.17 and 4.22 imply that for any point $\tau, \xi \in J \times G$ there is a unique maximal solution that is convenient to consider as a function $\varphi(t, \tau, \xi) : J \times J \times G \to G$ of three variables equal to the maximal solution x of (1). It is a common situation in applications that one is interested not in properties of one solution, but in a description of the family of solutions with all possible initial data as a whole. This type of problems constitute modern theory of differential equations and dynamical systems and motivates introducing the following notion.

Definition. p. 126. Transition map. The mapping $\varphi(t, \tau, \xi)$ defined above is called transition map.

In the case of autonomous systems there is no meaning in considering different initial times τ, because all solutions are functions of the time shift $t - \tau$. In this case we consider ransition mappings $\varphi(t, \xi) : J \times G \to G$ with $\varphi(t, \xi) = x(t)$ being the maximal solution of (2) with initial condition $x(0) = \xi$.

Example 4.33.

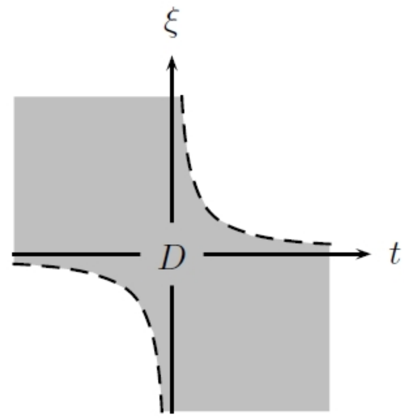
$G = \mathbb{R}$; $f : G \to \mathbb{R}$; $f(x) = x^2$; $\xi = 0$; $x(t) \equiv 0$.

$$\begin{aligned} \frac{dx}{dt} &= x^2; \quad \int \frac{dx}{x^2} = \int dt; \\ -\frac{1}{x} &= t + C \\ -\frac{1}{x} &= t - \frac{1}{\xi}; \quad -\frac{1}{x} = \frac{t\xi - 1}{\xi} \\ x &= \frac{\xi}{(1 - t\xi)} \end{aligned}$$

$\xi = 0$; $x(t) \equiv 0$. $\quad \xi > 0$, $I_\xi = (-\infty, 1/\xi)$. $\quad \xi < 0$, $I_\xi = (1/\xi, \infty)$

$$\varphi(t, \xi) = \frac{\xi}{(1 - t\xi)}; \quad D(\varphi) = \{(t, \xi) \in \mathbb{R} \times \mathbb{R}; \quad t\xi < 1\}$$

The domain D of φ is an open set. $\varphi(t, \xi)$ is continuous and even locally Lipschitz.



To make it easier to remember the meaning of theorems, in the case they do not have a historical names, we will supply them with meaningfull names, sometimes funny.

Proposition 4.4. Openness of maximal intervals.

Let $x : I \to G$ be a maximal solution to I.V.P. (1).The maximal interval I is relatively open in J (just open if $J = \mathbb{R}$).

It means that $I = J \cap O$ for some open set $O \subset \mathbb{R}$.

Theorem 4.8. p. 108. Existence of maximal solutions.

Every solution to (1) can be extended to a maximal solution.

Proof.

In the case when solutions are unique (for example f is locally Lipschitz with respect to x), one can build the maximal interval of existence just by as a union of domains for all extensions of a given solution. Because of the uniqueness of solutions, trajectories cannot make branches in this case and this construction leads to a unique maximal solution that at each time point t attains the value of one of the extensions defined at this time point. The uniqueness of solutions makes that this definition is consistent.

In the general case when trajectories can create branches, the union of extensions can have a tree like geometry, or even be an n-dimensional set. In this case the proof uses Zorn lemma to choose a maximal solution. It has an existence interval including all existence intervals of all extensions, but is possibly not unique.

The following technical lemma is the main tool in several arguments about maximal solutions.

Lemma 4.9. On extension of a bounded solution with closure in G to the boundary point of the open existence time interval.

Let $x : I \to G$ be a solution to (1) and denote $a = \inf I$; $b = \sup I$.

(1) If b is in J and not in I (I is open in the right end, the closure of the orbit $O_+ = \{x(t) : t \in [\tau, b)\}$ is a bounded (therefore compact) subset of G, then there is a solution $y : I \cup \{b\} \to G$ to (1) that is extension of x.

(2) a similar statement is valid for the "backward orbit" $O_- = \{x(t) : t \in (a, \tau]\}$ and extension of x to the left end point a.

Proof.

The following Corollary is a direct consequence of the Lemma 4.9 and Proposition 4.4 and gives a sufficient condition for a maximal solution to have an infinite maximal interval (if J is infinite) or a maximal interval "ifinite with respect to" J , which meaning is specified exactly below.

Corollary 4.10, p. 111. "Eternal life" of solutions enclosed in a compact.

If the "future" half - orbit $O_+ = \{x(t) : t \in I_{\max} \cap [\tau, \infty)\}$ of the maximal solution $x(t)$ is contained in a compact subset of G, then the corresponding maximal interval of existence $I_{\max}$ is infinite to the right (future) if $[\tau, \infty) \subset J$), or "infinite to the right with respect to J" meaning that the maximal solution exists on $[\tau, \infty) \cap I = [\tau, \infty) \cap J$ that is the whole part of J to the right of the initial time τ.

Similar statement is valid for the "backward orbit" $O_- = \{x(t) : t \in (a, \tau]\}$. If it is contained in a compact subset of G, then the corresponding maximal interval of existence $I_{\max}$ is infinite to the left (past) if $(-\infty, \tau] \subset J$ and is infinite to the left (past) "with respect to" J, that means that the maximal solution exists on $(-\infty, \tau] \cap I = (-\infty, \tau] \cap J$, that is the whole part of J to the left of the initial time τ.

If the whole orbit $O = \{x(t) : t \in I_{\max}\}$ of the maximal solution $x(t)$ is contained in a compact subset of G, then the corresponding maximal interval of existence $I_{\max} = J$ ($I_{\max} = \mathbb{R}$ if $J = \mathbb{R}$). It means that the maximal solution x exists both in the whole past and whole future for the equation.

Proof. The proof is easy to carry out by a contradiction argument that follows from the Lemma 4.9 and the fact that a maximal interval must be open (relatively to J).

The following Theorem describes the situation in a sense opposite to the previous Corollary 4.10. It describes the the behaviour of maximal solutions having bounded maximal interval $I_{\max}$ (if J is $\mathbb{R}$), and in the case when the interval J has bounded endpoints itself, describes maximal solution with maximal interval that is "bounded with respect to J", meaning that $\sup I_{\max} < \sup J$ or $\inf J < \inf I_{\max}$.

Theorem 4.11, p.112. "Short living" maximal solutions escape any compact.

Let $x : I \to G$ be a maximal solution to (1) with maximal interval of existence $I \subset J$ and assume that $I \neq J$. Denote $\alpha = \inf I$ and $\omega = \sup I$, both do not belong to I that must be open. Then either $\omega \in J \backslash I$ or $\alpha \in J \backslash I$.

1) In the first case $\omega \in J \backslash I$ for each compact $C \subset G$, there is an "escaping time moment" $\sigma \in I$, $\sigma < \omega$, such that $x(t)$ escapes C at time $\sigma : x(t) \notin C$ for all $t \in (\sigma, \omega)$.

This property can be further geometrically specified. If $G \neq \mathbb{R}^n$ the trajectory $x(t)$ tends to the boundary ∂G with $t \to \omega$ (if G is bounded) it can also tend infinity if G has "branches" going to infinity in $\mathbb{R}^n$. If $G = \mathbb{R}^n$, then $\|x(t)\| \to \infty, \quad as\ t \to \omega$.

$$
\begin{aligned}
\lim_{t\to\omega} \min\left\{\operatorname{dist}(x(t), \partial G), 1/\|x(t)\|\right\} &= 0, \quad for\ G \neq \mathbb{R}^n \qquad (4) \\
\|x(t)\| &\to \infty, \quad as\ t \to \omega, \quad for\ G = \mathbb{R}^n
\end{aligned}
$$

2) Similar statements are valid for the limits of $x(t)$ as $t \to \alpha$ for the maximal solution having maximal interval with the left end point "in the past" α belonging to J.

Proof.

We consider the case 1). The fact that the maximal solution must escape any compact C follows from the previous Corollary 4.10 by contradiction, because a solution that stays in a compact must have a maximal interval infinite to the right or $[\tau, \infty) \cap I = [\tau, \infty) \cap J$. It contradicts to the condition that $\omega \in J \backslash I$ that means that the given maximal $x(t)$ solution does not reach the maximal possible time in J.

A more sofisticated argument (missed in the course book) shows that there is a "last visit" time $\sigma < \omega$, such that $x(t)$ never enters C again after this time.

If G is bounded, one can choose a rising sequence of test compact sets $\{C_n\}_{n=1}^{\infty}$, $C_n \subset C_{n+1} \subset G$ like "blowing up ballons" tending to the boundary ∂G of G so that $\mathrm{dist}(C_n, \partial G) \to 0$ as $n \to \infty$. For each of these sets there is a time σ_n such that $x(t)$ leaves C_n and therefore has $\mathrm{dist}(x(t), \partial G) < \mathrm{dist}(C_n, \partial G)$ for $t > \sigma_n$. This construction proves the fact that $\mathrm{dist}(x(t), \partial G) \to 0$ as $t \to \omega$.

In the case of $G = \mathbb{R}^n$ one can choose a sequence of test compact sets $\{C_n\}_{n=1}^{\infty}$ as balls with centers in the origin and radii r_n tending to infinity with $n \to \infty$ leading together with the "escaping property" to conclusion that $\|x(t)\| \to \infty, \quad as\ t \to \omega$.

The third case with unbounded G with non-empty boundary ∂G can be proven by a combination of the above arguments.■

0.3