n	$f(z)$	R–H criterion
2	$a_0z^2 + a_1z + a_2$	$a_2 > 0,\ a_1 > 0$
3	$a_0z^3 + a_1z^2 + a_2z + a_3$	$a_3 > 0,\ a_1 > 0$ $a_1a_2 > a_0a_3$
4	$a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4$	$a_4 > 0,\ a_2 > 0,$ $a_1 > 0,$ $a_3(a_1a_2 - a_0a_3) > a_1^2a_4$

3.4 Two-Dimensional Linear Autonomous Systems

In this section we shall apply Theorem 3.3.6 to classify the behavior of the solutions of two-dimensional linear systems [H1]

$$\dot{x} = Ax,\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ \det A \neq 0 \qquad (3.10)$$

where a, b, c, d are real constants. Then $(0, 0)$ is the unique rest point of (3.10). Let λ_1, λ_2 be the eigenvalues of A, consider the following cases:

Case 1: λ_1, λ_2 are real and $\lambda_2 < \lambda_1$.

Let v^1, v^2 be unit eigenvectors of A associated with λ_1, λ_2 respectively. Then from (3.9), the general real solution of (3.10) is

$$x(t) = c_1e^{\lambda_1 t}v^1 + c_2e^{\lambda_2 t}v^2.$$

Case 1a (Stable node) $\lambda_2 < \lambda_1 < 0$.

Let L_1, L_2 be the lines generated by v^1, v^2 respectively. Since $\lambda_2 < \lambda_1 < 0$, $x(t) \approx c_1e^{\lambda_1 t}v^1$ as $t \to \infty$ and the trajectories are tangent to L_1. The origin is a stable node (see Fig. 3.1).

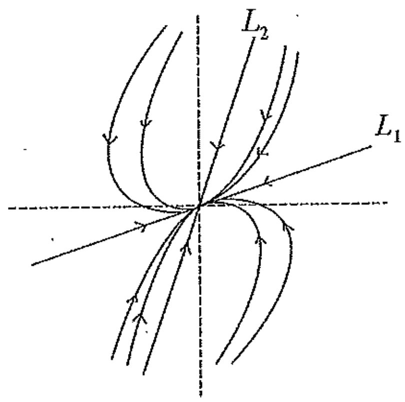

Fig. 3.1

Case 1b (Unstable node) $0 < \lambda_2 < \lambda_1$.
Then $x(t) \approx c_1 e^{\lambda_1 t} v^1$ as $t \to \infty$. The origin is an unstable node (see Fig. 3.2).

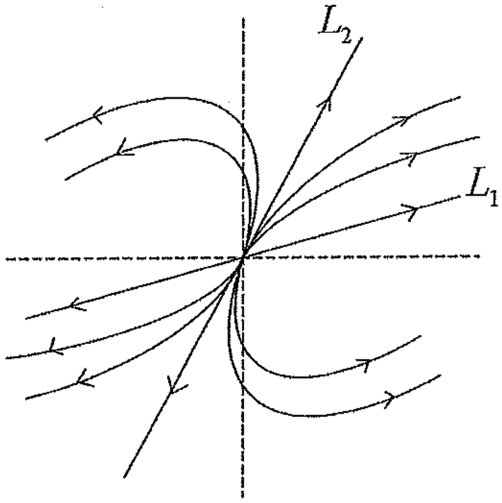

Fig. 3.2

Case 1c (Saddle point) $\lambda_2 < 0 < \lambda_1$. In this case, the origin is called a saddle point and L_1, L_2 are called unstable manifold and stable manifold of the rest point $(0,0)$ respectively (see Fig. 3.3).

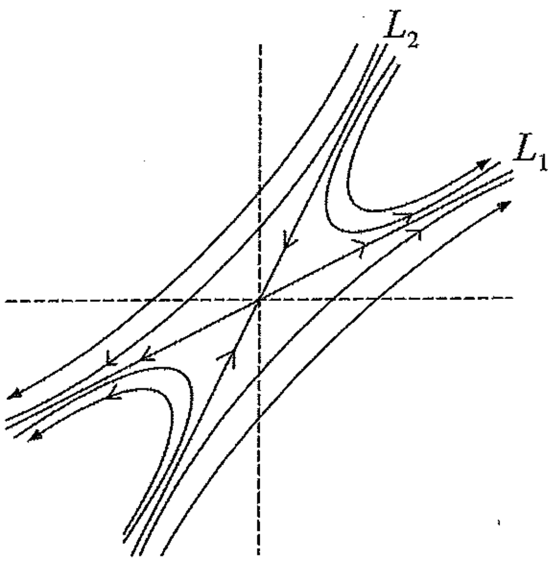

Fig. 3.3

Case 2: λ_1, λ_2 are complex.
Let $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ and $v^1 = u + iv$ and $v^2 = u - iv$ be

complex eigenvectors. Then

$$x(t) = ce^{(\alpha+i\beta)t}v^1 + \bar{c}e^{(\alpha-i\beta)t}\overline{v^1} = 2Re\ \left(ce^{(\alpha+i\beta)t}v^1\right).$$

Let $c = ae^{i\delta}$. Then

$$x(t) = 2ae^{\alpha t}\left(u\cos(\beta t+\delta) - v\sin(\beta t+\delta)\right).$$

Let U and V be the lines generated by u, v respectively.

Case 2a (Center) $\alpha = 0,\ \beta \neq 0$. The origin is called a center (see Fig. 3.4).

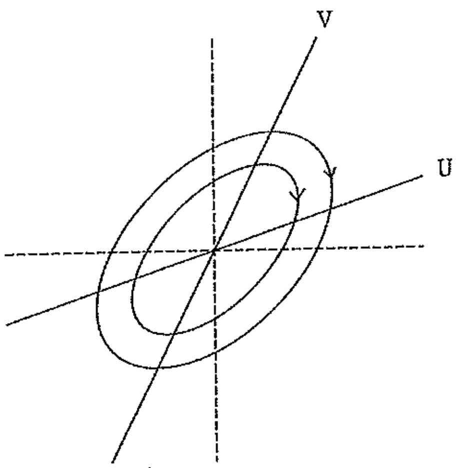

Fig. 3.4

Case 2b (Stable focus, spiral) $\alpha < 0,\ \beta \neq 0$. The origin is called a stable focus or stable spiral (see Fig. 3.5).

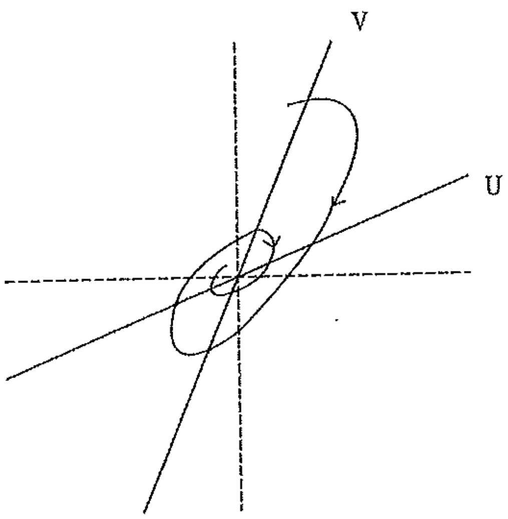

Fig. 3.5

Case 2c (Unstable focus, spiral) $\alpha > 0$, $\beta \neq 0$. The origin is called an unstable focus or unstable spiral (see Fig. 3.6).

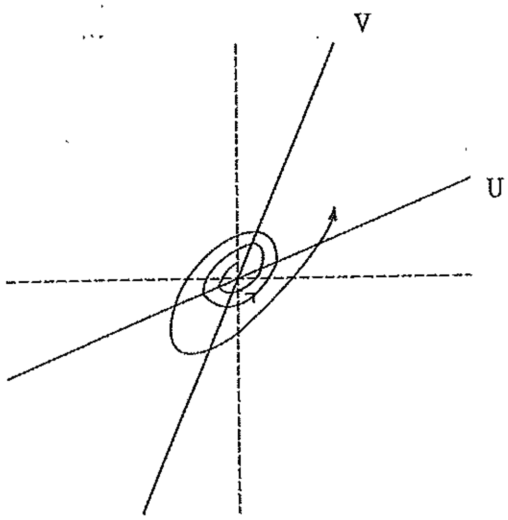

Fig. 3.6

Case 3 (Improper nodes) $\lambda_1 = \lambda_2 = \lambda$

Case 3a: There are two linearly independent eigenvectors v^1 and v^2 of the eigenvalue λ. Then,

$$x(t) = \left(c_1 v^1 + c_2 v^2\right) e^{\lambda t}.$$

If $\lambda > 0$ ($\lambda < 0$) then the origin 0 is called an unstable (stable) improper node (see Fig. 3.7).

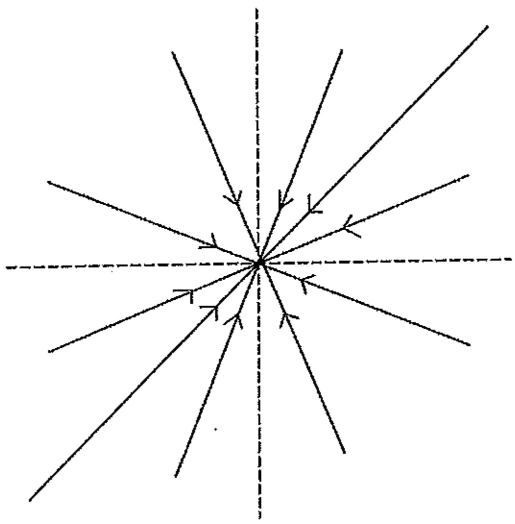

Fig. 3.7

Case 3b: There is only one eigenvector v^1 associated with eigenvalue λ. Then from (3.9) . v^2 - generalized eigenvector.

$$x(t) = (c_1 + c_2 t)\, e^{\lambda t} v^1 + c_2 e^{\lambda t} v^2$$

where v^2 is any vector independent of v^1 (see Fig. 3.8).

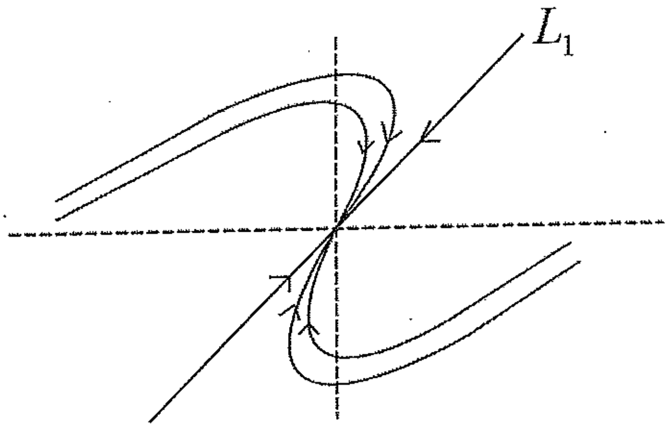

Fig. 3.8