Home assignment 1 for the course ODE and mathematical modelling MVE162 in year 2019.
Two models of biological competition.

The modeling assignment is obligatory and consists of a theoretical part that requires some mathematical reasoning and an implementation part including writing a simple Matlab code solving an ODE, graphical output, analysis of numerical solutions and conclusions.

Each of you must write an own report but you are encouraged to work in small groups of 2-3 people discussing theoretical and programming problems.

The report must we written as a small scientific article that a person who did not study the course can understand. It must include: ***1) theoretical argumentation, with necessary references 2) numerical results with graphical output and 3) interpretation of results, and names of all group members. Second year bachelor students attend a lecture on scientific writing by Elin Götmark on the 4-th of April and will have possibility to discuss the scientific writing aspects of the preliminary version of reports in Swedish.***

Send final reports and Matlab codes with clear comments by e-mail to Alexei Heintz, heintz(at)chalmers.se and to Elin Götmark, *elin(at)chalmers.se. Deadline for delivering reports is Friday 3/5.*

Grades for your reports on two home assignments will contribute 16% each to the final marks for the course.

Logistic equation and two species competition model.

Let $x_i(t), i = 1, 2.$ be populations of two species. Each of the species grows with intrinsic growth rate r_i in case when infinite resources are available: $x_i' = r_i x_i$, $r_i > 0$.

Limited resources lead to competition within the population and a limited growth rate for large size of the population: $r_i(1 - \frac{x_i}{K_i})$. This model is called the logistic equation:

$$x_i' = r_i x_i \left(1 - \frac{x_i}{K_i}\right) \tag{1}$$

The competition between different species leads a decrease in each population with the decreasing rate proportional to the competitor population size: $-\alpha_1 x_2$ for the population x_1 and $-\alpha_2 x_1$ for the population x_2 with competition coefficients $\alpha_1 > 0$ and $\alpha_2 > 0$. The corresponding system of equations describes the evolution of two competing species:

$$\begin{aligned} x_1' &= r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 \\ x_2' &= r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - \alpha_2 x_2 x_1 \end{aligned} \tag{2}$$

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You should not think of the numbered list in the working plan below as sections of your paper. Instead, think about how best to organize your paper according to the lecture on scientific writing.

Working plan and questions for the assignment N1

1. Describe all qualitatively different solutions to the equation (1) without solving the equation analytically.

2. Illustrate your theoretical conclusions about solutions of (1) by a graph with several representative curves for different initial data and give a biological interpretation of your mathematical conclusions.

3. Show that solutions I.V.P. for (2) starting from positive initial data for number of species $x_1(0) > 0$, $x_2(0) > 0$, will always have positive components for $t > 0$.

4. Prove that all solutions to (2) corresponding to $x_1(0) \geq 0$, $x_2(0) \geq 0$ never go to infinity.

5. Show that the maximal existence interval for solutions to the system (2) starting at $t = 0$ with $x_1(0) \geq 0$, $x_2(0) \geq 0$, is $\mathbb{R}_+$ (a useful argument was introduced in lecture notes in the proof to the Theorem 5.27).

6. Consider isoclines of the system (2) that are lines where $x_1' = 0$ or $x_2' = 0$. Their intersections are stationary (equilibrium) points. Describe all possible qualitatively different positions of isoclines and equilibrium points for various relations between the parameters $\frac{r_1}{\alpha_1}$, K_1 and $\frac{r_2}{\alpha_2}$, K_2 (there are four distinct cases). Characterize them analytically. Draw pictures with isoclines illustrating each case.

7. Investigate stability properties of all equilibrium points of (2) for each of these qualitatively different positions of isoclines. Do it by applying the Grobman-Hartman theorem and the criterion for classification of phase portraits around equilibrium points for linear systems in plane by the determinant and the trace of the systems matrix.

8. Draw pictures with phase portraits together with isoclines for each of the cases above, and make conclusions about different scenarios of the evolution for (2).

9. Give a biological interpretation to each of the four possible scenarios of evolution by (2).

10. Is there a qualitative difference between the behavior of solutions to the logistic equation and to the two species competition model when $t \to \infty$? How this difference depends on α_1 and α_2 if they are small or large (with other parameters fixed)?

Hints to Matlab implementation

1. Write two Matlab codes: one that can solve a scalar ODE and one that can solve a system of two ODEs. Let both programs choose several initial data points and draw corresponding integral curves both as functions of time. In the case of the system of two equations the code must draw trajectories of $(x_1(t), x_2(t))$ in the phase plane of (x_1, x_2) in a separate figure. You can use function *plot* for graphics. In the case of the system of two equations the function *ginput* in Matlab can be used (not obligatory) to choose a point of initial data from the plane of (x_1, x_2).