1 Banach's contraction principle. Picard-Lindelöf theorem.

We consider in this chapter the theorem by Picard and Lindelöf about existence and uniqueness of solutions to the initial value problem to the system of differential equations in the form

$$x'(t) = f\left(t, x(t)\right) \tag{1}$$

$$x(\tau) = \xi \tag{2}$$

Here $f : J \times G \to \mathbb{R}^n$ is a vector valued function continuous with respect to time variable t and space variable x. J is an interval, G is an open subset of $\mathbb{R}^n$.

One can reformulate the I.V.P. (1),(2) in the form of the integral equation

$$x(t) = \xi + \int_\tau^t f\left(s, x(s)\right) ds \tag{3}$$

If f is continuous, then these two formulations are equivalent by the Newton-Leibnitz theorem.

Fixed points of operators.

Consider a vector space X with a subset $C \subset X$ and an operator $K : C \to C$.

Definition

A point $\overline{x} \in C$ is called the fixed point of the operator K if

$$K(\overline{x}) = \overline{x} \tag{4}$$

A general idea behind the analysis of many types of equations is to formulate them as a fixed point problem.

Consider the right hand side of the integral equation (3) as an operator

$$K(x)(t) \stackrel{\text{def}}{=} \xi + \int_{\tau}^{t} f\left(s, x(s)\right) ds$$

acting from the vector space of continuous functions $C(I)$, where $I \subset J$ is a closed interval including τ .

The expression $\|x\|_{C(I)} = \sup_{t \in I} \|x(t)\|$ defines a norm on the space $C(I)$ because it satisfies the triangle inequality and we know that uniformly convergent sequences of continuous functions on the compact converge to continuous functions. This space is even complete in the sense that Cauchy sequences of functions in $C(I)$ converge uniformly to continuous functions. It means that if the sequence $\{x_n\} \in C(I)$ has the Cauchy property

$$\|x_m - x_n\|_{C(I)} = \sup_{t \in I} \|x_m(t) - x_n(t)\|_{C(I)} \underset{m,n \to \infty}{\longrightarrow} 0$$

then there is a continuous function $\overline{x} \in C(I)$ such that $x_n \underset{n \to \infty}{\longrightarrow} \overline{x}$ uniformly on I, or that is the same, $\|x_n - \overline{x}\|_{C(I)} \underset{n \to \infty}{\longrightarrow} 0$.

We call a normed vector space a Banach space if it is complete with respect to it's norm. So the space $C(I)$ is a Banach space.

Remark.

We point out for convenience that different norms are used through out the text. Notation $\|\|$ means usual euclidean norm in $\mathbb{R}^n$. For a Banach space X the notation $\|x\|_X$ means the norm in the space X.

The operator K defined above, acts from $C(I)$ to itself. It makes that the I.V.P. above can be considered as a fixed value problem (**4**) on $C(I)$ or on some subset of it.

A classical theorem that guarantees the existence and uniqueness of fixed points to operators in Banach and more generally in metric spaces, is Banach's contraction principle.

Definition. Operator $K : A \to A$, where $A \subset X$, and X is a Banach space, is called contraction on A if there is a constant $0 < \theta < 1$ such that for any $x, y \in A$

$$\|K(x) - K(y)\|_X \le \theta \|x - y\|_X$$

Banach's contraction principle.

Let A be a non-empty closed subset of a Banach space X and $K : A \to A$ be a contraction operator with contraction constant θ.Then there is a unique fixed point $\overline{x} \in A$, to K such that $K\overline{x} = \overline{x}$ such that

$$\|K^n(x_0) - \overline{x}\|_X \le \frac{\theta^n}{1-\theta} \|K(x_0) - \overline{x}\|_X$$

for arbitrary $x_0 \in A$. Here $K^n(x_0) = K(K(...K(x_0))...)$ is the operator K applied to itself n times.

Proof (not required at the exam) is based on showing that sequential approximations x_n defined by the equations

$$\begin{aligned} x_1 &= K(x_0) \\ x_{n+1} &= K(x_n) \end{aligned}$$

with an arbitrary initial approximation $x_0 \in A$, converge to some $\overline{x} \in A$ that is the unique fixed point of K in A.

Picard-Lindelöf theorem.

Here $f : J \times G \to \mathbb{R}^n$ is a vector valued function continuous in $J \times G$. J is an interval, G is an open subset of $\mathbb{R}^n$. Let in addition suppose that f is Lipschitz continuous with respect to the second argument with the Lipschitz constant $L > 0$:

$$\|f(t,x) - f(t,y)\| \le L \|x - y\|, \forall x, y \in G$$

(We could suppose a weaker condition that this Lispchitz property is only local, but will not do it because it would make the proof just slightly longer without changing main ideas).

Then for any $(\tau, \xi) \in J \times G$ the initial value problem

$$\begin{aligned} x' &= f(x,t) \\ x(\tau) &= \xi \end{aligned}$$

has a unique solution on some time interval including τ. $\square$

Remark. This local solution can always be extended to a unique maximal solution. We considered maximal extensions earlier in the course.

Proof to the Picard-Lindelöf theorem.

The proof is based on using the integral form of the I.V.P.

$$x(t) = \xi + \int_{\tau}^{t} f\left(s, x(s)\right) ds$$

and applying Banachs contraction principle to it. We use the Banach space of continuous functions $x : I \to \mathbb{R}^n$on some compact interval I.

The application of Banach's principle here consists of two steps. One is to find a time interval I_1 and a subset $A \subset C(I_1)$ such that the operator K maps A to itself: $K : A \to A$.

Another one is to find a time interval I_2 such that the contractness property for the operator

$$K(x)(t) = \xi + \int_{\tau}^{t} f\left(s, x(s)\right) ds$$

would be valid on a subset of $C(I_2)$. Finally we will choose the smallest of I_1 and I_2 for both properties to be valid and will conclude the result.

We consider here the case when an interval $[\tau, \tau + T] \in J$, $T > 0$ and try to find a solution on this time interval (or possibly on a shorter time interval $[\tau, \tau + \Delta]$ with $\Delta < T$). Considering a time interval backwords in time is similar. Choose a closed ball $\overline{B(\xi, \delta)} = \{x : \|x - \xi\| \leq \delta\}$ such that

$\overline{B(\xi,\delta)} \in G$.

The function $f(t,x)$ is continuous on the compact set $V = [\tau, \tau + T] \times \overline{B(\xi,\delta)}$ in $\mathbb{R}^{n+1}$and therefore

$$M = \sup_{(t,x)\in V} \|f(t,x)\| < \infty$$

We are going to estimate $\|K(x)(t) - \xi\|$ and choose the length T of the time interval $[\tau, \tau + T]$ in such a way that for any $x(t) \in \overline{B(\xi,\delta)}$ for $t \in [\tau, \tau + T]$, it follows that $\|K(x(t)) - \xi\| \leq \delta$ for $t \in [\tau, \tau + T]$.

It would imply that $\sup_{t\in[\tau,\tau+T]} \|K(x)(t) - \xi\| = \|K(x) - \xi\|_{C([\tau,\tau+T])} \leq \delta$ for $\|x - \xi\|_{C([\tau,\tau+T])} \leq \delta$. We start with proving the first inequality:

$$\|K\left(x\right)\left(t\right) - \xi\| = \left\|\int_{\tau}^{t} f\left(s, x(s)\right) ds\right\| \leq \int_{\tau}^{t} \|f\left(s, x(s)\right)\| \, ds \leq TM$$

We observe that choosing $T < \delta/M$ we get that $\|K\left(x\right)\left(t\right) - \xi\| \leq \delta$ for $t \in [\tau, \tau + T]$ and taking supremum of the left hand side over $t \in [\tau, \tau + T]$ arrive to

$$\|K(x) - \xi\|_{C([\tau,\tau+T])} \leq \delta$$

It means that the operator K maps the closed ball A in $C([\tau, \tau+T])$ with $T < \delta/M$, defined by the inequality $\|x - \xi\|_{C([\tau,\tau+T])} \leq \delta$ into itself:

$$K : A \to A$$

Now we check conditions (again the length of the time interval) such that the operator K would be contraction on the set A with a suitably adjusted time interval T. Consider first the difference $\|K\left(x\right)\left(t\right) - K\left(y\right)\left(t\right)\|$, for $t \in [\tau, \tau + T]$, apply the triangle inequality, the Lipschitz property of the function f , and estimate the integral by the length of the interval times maximum of the function under it.

$$
\begin{aligned}
\|K(x)(t)-K(y)(t)\| &= \left\|\int_\tau^t f(s,x(s))-f(s,y(s))\,ds\right\| \leq \\
\int_\tau^t \|f(s,x(s))-f(s,y(s))\|\,ds &\leq L\int_\tau^t \|x(s)-y(s)\|\,ds \leq \\
LT\sup_{t\in[\tau,\tau+T]}\|x(s)-y(s)\|\,ds &= LT\,\|x-y\|_{C([\tau,\tau+T])}
\end{aligned}
$$

Calculating supremum over $t \in [\tau, \tau+T]$ of the left hand side we arrive to the inequality

$$
\|K(x)-K(y)\|_{C([\tau,\tau+T])} \leq LT\,\|x-y\|_{C([\tau,\tau+T])}
$$

It implies that choosing $T < 1/L$ we get the contraction property.

$$
\|K(x)-K(y)\|_{C([\tau,\tau+T])} \leq \theta\,\|x-y\|_{C([\tau,\tau+T])}\,, \quad 0<\theta<1
$$

Now choosing the time interval $T < \min(1/L,\ \delta/M)$ we conclude that the operator K maps the closed ball A in $C([\tau,\tau+T])$ defined by

$$
A = \left\{x \in C([\tau,\tau+T]),\ \|x-\xi\|_{C([\tau,\tau+T])} \leq \delta\right\}
$$

into itself: $K : A \to A$ and that K is a contraction on A: $\|K(x)-K(y)\|_{C([\tau,\tau+T])} \leq \theta\,\|x-y\|_{C([\tau,\tau+T])}$, $\theta < 1$, for any $x, y \in A$.

By the Banach contraction principle K has for $T < \min(1/L,\ \delta/M)$ a unique fixed point $\overline{x}$ in A that is the solution to the integral equation (3) and to the original initial value problem.■

Exercise.

Consider the following (nonlinear!) operator

$$
K(x)(t) = \int_0^2 B(t,s)\,[x(s)]^2\,ds + g(t),
$$

acting on the Banach space $C([0,2])$ of continuous functions with norm $\|x\|_{C([0,2])} = \|x\|_C = \sup_{t\in[0,2]} |x(t)|$. Here $B(t,s)$ and $g(t)$ are continuous functions and $|B(t,s)| < 0.5$ for all $t, s \in [0,2]$. Estimate the norm $\|K(x) - K(y)\|_{C([0,2])}$ for the operator $K(x)(t)$. Find requirements on the function $g(t)$ such that Banach's contraction principle implies that $K(x)(t)$ has a fixed point.

Solution.

Banach's contraction principle. Let B be a nonempty closed subset of a Banach space X and let the non-linear operator $K : B \to B$ be a contraction.

$$\|K(x) - K(y)\|_X \le \theta \|x - y\|_X , \theta < 1$$

Then K has a fixed point $\overline{x} = K(\overline{x})$ such that

$$\|K^n(x_0) - \overline{x}\|_X \le \frac{\theta^n}{1-\theta}$$

for any $x_0 \in B$. Here $K^n(x_0) = (K(K(...K(x_0)...))$ is the n -fold superposition of the operator K with itself.

We like to have the estimate $\|K(x) - K(y)\| \le \theta \|x - y\|$ for x, y in some closed subset B of $C([0,2])$.

$|K(x) - K(y)| \le \left| \int_0^2 |B(t,s)| \left| [x(s)]^2 - [y(s)]^2 \right| ds \right|$

$= \left| \int_0^2 |B(t,s)| \, |x(s) - y(s)| \, |x(s) + y(s)| \, ds \right| \overset{\text{taking} \quad \sup\limits_{t,s\in[0,2]}}{\le}$

$\int_0^2 ds \sup\limits_{t,s\in[0,2]} |B(t,s)| \sup\limits_{s\in[0,2]} |x(s) - y(s|) \left(\sup\limits_{s\in[0,2]} |x(s)| + \sup\limits_{s\in[0,2]} |y(s)| \right) =$

$2 \cdot 0.5 \|x - y\|_C (\|x\|_C + \|y\|_C) = \|x - y\|_C (\|x\|_C + \|y\|_C)$

We can choose a ball $B \subset C([0,2])$ such that for any x, $y \in B$ it follows $\|x\|_C + \|y\|_C \le \theta < 1$, for example B can be taken as a set of functions with $\|x\|_C \le \theta/2$. On this set K will be a contraction because $\|K(x) - K(y)\|_C \le \theta \|x - y\|_C$, $\theta < 1$.

To apply Banach's principle we need also that K maps B into itself, namely that $\|K(x)\|_C \le \theta/2$ for $\|x\|_C < \theta/2$.

It gives a requirement on function $g(t)$. Estimate the operator $K:$

$\|K(x)\|_C \leq \sup\limits_{t\in[0,2]} \left| \int_0^2 B(t,s) \left([x(s)]^2\right) ds \right| + \sup\limits_{t\in[0,2]} |g(t)| \leq \|x\|_C^2 + \|g\|_C$

If $\|x\|_C < \theta/2$ then we like to have that $\|K(x)\|_C \leq \theta/2$ that follows if $\|K(x)\|_C \leq \theta^2/4 + \|g\|_C \leq \theta/2$

It is satisfied if $\|g\|_C \leq \theta/2 - \theta^2/4 = \theta/2(1-\theta/2)$. Therefore for $\|g\|_C \leq \theta/2(1-\theta/2)$ the operator K has a unique fixed point in the ball $B : \|x\|_C \leq \theta/2$.