Exercises on linear ODE with periodic coefficients.

1. Find the characteristic (Floquet) multiplicator for the scalar linear equation with periodic coefficient: **(4p)**

$$x' = (a + \sin^2 t)x$$

Find also those values of the parameter a that imply that all solutions tend to zero with $t \to +\infty$.

2. Calculate monodromy matrix and Floquet exponents for the 2-dim system

$$x' = a(t)Ax$$

where $a(t)$ is a scalar periodic function with period T and A is a constant real 2×2 matrix. Discuss conditions implying that all solutions tend to zero or stay bounded with $t \to +\infty$.

Hint: make a change of time variable $t \to \tau = \int_{t_0}^{t} a(s)ds$.

3. Compute the monodromy matrix for the system with the following periodic matrix $A(t)$ with period 1.

$$A(t) = \begin{cases} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} = A_1, & 0 \leq t < 1/2 \\ \\ \begin{bmatrix} \alpha & 0 \\ 1 & \alpha \end{bmatrix} = A_2, & 1/2 \leq t < 1 \end{cases}$$

Hint: use the same idea as in the Exercise 2 and combine explicit formulas for fundamental matrices on subintervals where $A(t)$ is a constant matrix.

Some solutions

1. Find the characteristic multiplicator for the scalar linear equation with periodic coefficient: **(4p)**

$$x' = (a + \sin^2 t)x$$

1. The characteristic multiplicator is eigenvalue of the monodromy matrix denoted by $M = e^{TR}$ in the course book, where T is the period of the right hand side in the equation. One builds a monodromy matrix (it will be a number in our case with one scalar equation) of solutions to initial value problems with initial data $x(0)$ that are standard basis vectors in R^n calculated in the time point T - equal to the period of the right hand side.In our case we have just one scalar equation, so the monodromy matrix will be a number. We find the value of the solution to I.V.P. to the given equation with initial data $x(0) = 1$ at the time $t = 2\pi$ that is a period of the right hand side in our case. The equation is linear, so the solution is found with help of a primitive function of the coefficient:

   $P(t) = \int_0^t (a + \sin^2 s)ds = \frac{1}{2}t + at - \frac{1}{4}\sin 2t.$

   $x(t) = \exp(P(t))x(0) = \exp\left(\frac{1}{2}t + at - \frac{1}{4}\sin 2t\right) x(0).$

   The monodromy "matrix" in our case is the value of the solution $x(t)$ in $t = 2\pi$ such that $x(0) = 1$.

   $x(2\pi) = \exp\left(\frac{1}{2}2\pi + a2\pi\right) = \exp\left(\pi(1 + 2a)\right).$

   The characteristic multiplicator is the same number: $\exp\left(\pi(1 + 2a)\right)$. Solutions will tend to zero in the case $a < -1/2$, that makes $\exp\left(\pi(1 + 2a)\right) < 1$.

3. Answer:

The monodromy matrix $\Phi(p,0) = \Phi(1,0)$ is expressed as $\Phi(1,0) = \exp((\alpha(1-1/2)\, A_2)\exp((\alpha/2)\, A_1)$

1. Here $\exp(tA_1) = \exp(\alpha t)\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, $\exp(tA_2) = \exp(\alpha t)\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$

   We derive an explicit expression for $\Phi(1,0) = \exp(\alpha)\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

   $=\exp(\alpha)\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix}$,

   $\det\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix} = 1$; $Tr\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix} = 2.25.$

   characteristic polynomial $p(\lambda) = \lambda^2 - \frac{9}{4}\lambda + 1$

   eigenvalues: $\lambda_1 = \frac{9}{8} - \sqrt{\left(\frac{9}{8}\right)^2 - 1} = \frac{9}{8} - \frac{1}{8}\sqrt{17} > 0$, $\lambda_2 = \frac{1}{8}\sqrt{17} + \frac{9}{8} > 0$ and are simple.

   Condition for boundedness of all solutions is $\exp(\alpha)\,|\lambda_2| \le 1$ or $\exp(\alpha)\frac{1}{8}\left(\sqrt{17}+9\right) \le 1$ because λ_2 is larger in absolute value.

   or it can be reforemulated by taking logarithm of left and right hand sides as $\alpha \le \ln(8) - \ln(\sqrt{17}+9) \approx -0.494\,93$.

   All solutions will tend to zero if and only if $\alpha < \ln(8) - \ln(\sqrt{17}+9) \approx -0.494\,93$