

1 General properties of ω -limit sets and Salle's invariance principle and it's applications to asymptotic stability §5.2

Example. An elementary introduction to LaSalle's invariance principle.

We like to investigate stability of equilibrium point in the origin for the system

$$\begin{aligned}x_1' &= x_2 \\x_2' &= -x_1 - x_2^3\end{aligned}$$

Using the simple test function $V(x_1, x_2) = x_1^2 + x_2^2$ we observe that it is a Lyapunov function for the system:

$$V_f(x_1, x_2) = \nabla V \cdot f(x_1, x_2) = -2x_2^4 \leq 0$$

and the origin is a stable equilibrium point. On the other hand V is not a strong Lyapunov function, because $V_f(x_1, x_2) = 0$ not only in the origin, but on the whole x_1 - axis where x_2 is zero. We could try to find a more sophisticated

On the other hand considering the vector field of velocities of this system on the x_1 - axis, we observe that they are crossing the x_1 - axis (even are orthogonal to it in this particular example) in all points except the origin. It means that all trajectories of the system cross and immediately leave the x_1 - axis that is the line where $V_f(x_1, x_2) = 0$ (the Lyapunov function is not strong). This observation shows that in fact the Lyapunov function $V(\varphi(t, \xi))$ is strictly monotone along trajectories $\varphi(t, \xi)$ everywhere except discrete time moments, when $\varphi(t, \xi)$ crosses the x_1 - axis. More explicitly in polar coordinates r and θ :

$$(r^2)' = -2r^4 \sin^4 \theta$$

We can therefore conclude that $V(\varphi(t, \xi)) \searrow 0$ as $t \rightarrow \infty$ and therefore,

the origin is asymptotically stable equilibrium of this system of equations.

One can also get a more explicit picture of this dynamics by looking on the equation for the polar angle θ :

$$\begin{aligned} \left(\frac{x_2}{x_1}\right)' &= (\tan(\theta))' = \frac{\theta'}{\cos^2(\theta)} \\ \frac{x_2'x_1 - x_1'x_2}{x_1^2} &= \frac{(-x_1 - x_2^3)x_1 - (x_2)x_2}{x_1^2} \\ &= \frac{(-x_1^2 - x_2^2 - x_1x_2^3)}{x_1^2} = \frac{-r^2 - \cos\theta \sin^3\theta r^4}{r^2 \cos^2\theta} \end{aligned}$$

$$\begin{aligned} \theta' &= -1 - \cos\theta \sin^3\theta r^2 = -1 - \frac{(\sin 2\theta \sin^2\theta) r^2}{2} \\ &= -1 - \frac{\sin 2\theta(1 - \cos 2\theta)r^2}{4} < 0, \quad r < 2 \end{aligned}$$

We see that the trajectories tend to the origin going (non-uniformly) as spirals clockwise around the origin.

This example demonstrates the main idea with applications of the LaSalle invariance principle to asymptotic stability of equilibrium points.

Proposition. Simple version of LaSalle's invariance principle.
Theorem 5.15. p. 183

We find a simple "weak" Lyapunov function $V_f(z) \leq 0$ for $z \in U$ in the domain $U \subset G$, $0 \in U$. This fact implies stability of the equilibrium. Then we check what happens on the set $V_f^{-1}(0)$ where $V_f(z) = 0$. If the set $V_f^{-1}(0)$ contains no other orbits except the equilibrium point, this equilibrium point in the origin must be asymptotically stable.

It's attracting region W can be chosen as a bounded open positively invariant set $W \subset U$ including the equilibrium.

Any trajectory starting in W will have positive orbit with a compact closure. We need this property for applying LaSalle's invariant principle describing ω -limit sets for positive orbits of solutions to ODEs.

The next theorem gives a simple criterion for having the whole space as the domain of attraction for an asymptotically stable equilibrium point.

Theorem 5.22, p. 188. On global asymptotic stability

Assume that $G = \mathbb{R}^n$. Let the hypothesis of the orem 5.15 hold with $U = G = \mathbb{R}^n$. If in addition the Lyapunov function V is radially unbounded:

$$V(z) \rightarrow \infty, \quad \|z\| \rightarrow \infty$$

then the origin 0 is globally stable equilibrium that means that all solutions $\|\varphi(t, \xi)\| \rightarrow 0$, as $t \rightarrow \infty$.

Exercise.

Show that all trajectories of the system

$$\begin{aligned}x' &= y \\y' &= -x - (1 - x^2)y\end{aligned}$$

that go through points in the domain $\|x\| < 1$, tend to the origin. Or by other words, show that the origin is an asymptotically stable equilibrium and that the circle $\|x\| < 1$ is it's domain of attraction.

Exercise 5.17

The aim of this exercise is to show that the condition of radial unboundedness in Theorem 5.22 is essential.

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by

$$f(z) = f(z_1, z_2) = \begin{cases} (-z_1, z_2) & \text{if } z_1^2 z_2^2 \geq 1 \\ (-z_1, 2z_1^2 z_2^3 - z_2) & \text{if } z_1^2 z_2^2 < 1. \end{cases}$$

Define $V: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$V(z) = V(z_1, z_2) = z_1^2 + \frac{z_2^2}{1 + z_2^2}.$$

- (a) Show that the equilibrium 0 of (5.1) is asymptotically stable.
- (b) Show that the equilibrium 0 is *not* globally asymptotically stable.
- (c) Show that V is not radially unbounded.

More general formulation and a proof of the LaSalle's invariance principle use some general properties of transition mappings, and ω - limit sets. We collect them here and give some comments about their proofs.

We consider I.V.P. and corresponding transition map $\varphi(t, \xi)$ for the system

$$\begin{aligned}x' &= f(x) \\x(0) &= \xi\end{aligned}$$

with $f : G \rightarrow \mathbb{R}^n$, G - open, $G \subset \mathbb{R}^n$, f is locally Lipschitz, $\xi \in G$.

1.1 Main theorem on the properties of limit sets.

The next theorem on the properties of ω - limit sets collects properties of ω - limit sets valid for systems of any dimension, in contrast with the Poincare - Bendixson theorem and its generalization, that gives a

description of ω - limit sets only for systems in plane, or on 2-dimensional manifolds.

Main theorem on the properties of limit sets. Theorem 4.38, p.143

We keep the same limitations and notations for the autonomous system as above.

Let $\xi \in G$. If the closure of the positive semi-orbit $O^+(\xi)$ is compact and contained in G , then $\mathbb{R}_+ \subset I_\xi$ and the ω - limit set $\Omega(\xi) \subset G$ is

- 1) non-empty
- 2) compact
- 3) connected
- 4) invariant (both positively and negatively): namely for any ω - limit point $\eta \in \Omega(\xi)$, the maximal interval $I_\eta = \mathbb{R}$ for initial data in η , and $\varphi(t, \eta) \in \Omega(\xi)$ for all $t \in \mathbb{R}$.
- 5) $\varphi(t, \xi)$ approaches $\Omega(\xi)$ as $t \rightarrow \infty$:

$$\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, \xi), \Omega(\xi)) = 0$$

Remark

The most interesting statement in the theorem is statement 4). It means that ω - limit sets consist of orbits of solutions to the system. Taking a starting point η on the limit set $\Omega(\xi)$ we get a trajectory $\varphi(t, \eta)$ that stays within this set $\Omega(\xi)$ infinitely long both in the future and in the past.

A simple tool to satisfy conditions in this theorem is to find a compact positively invariant set for the system that contains the point ξ . It can be done using one of two methods discussed earlier.

Proofs of statements in the theorem are based on: general properties of compact sets for 1) ,2), simple contradiction arguments and the definition of the limit sets for 3) and the translation property of the transition mapping $\varphi(t, \xi)$, together with continuity of $\varphi(t, \xi)$ for 4), a contradiction argument together with the definition of ω - limit sets.

LaSalle's invariance principle

We formulate now LaSalle's invariance principle that generalizes ideas that we discussed in the introductory example and gives a handy instrument for localizing ω - limit sets of non-linear systems in arbitrary dimension.

Theorem 5.12, p.180

Assume that f is locally Lipschitz as before and let $\varphi(t, \xi)$ denote the flow generated by the corresponding system

$$x' = f(x)$$

Let $U \subset G$ be non-empty and open. Let $V : U \rightarrow \mathbb{R}$ be continuously differentiable and such that $V_f(z) = \nabla V \cdot f(z) \leq 0$. for all $z \in U$. If $\xi \in U$ is such that the closure of the semi-orbit $O^+(\xi)$ is compact and is contained in U , then $\mathbb{R}_+ \subset I_\xi$ (maximal existence interval for ξ) and $\varphi(t, \xi)$ approaches as $t \rightarrow \infty$ the largest invariant set contained in $V_f^{-1}(0)$ that is the set where $V_f(z) = 0$.

Proof.

Proof given in the solution of Exercise 5.9, on p. 312.

Exercise 5.9

Set $x(\cdot) := \varphi(\cdot, \xi)$. By continuity of V and compactness of $\text{cl}(O^+(\xi))$, V is bounded on $O^+(\xi)$ and so the function $V \circ x$ is bounded. Since $(d/dt)(V \circ x)(t) = V_f(x(t)) \leq 0$ for all $t \in \mathbb{R}_+$, $V \circ x$ is non-increasing. We conclude that the limit $\lim_{t \rightarrow \infty} V(x(t)) =: \lambda$ exists and is finite. Let $z \in \Omega(\xi)$ be arbitrary. Then there exists a sequence (t_n) in \mathbb{R}_+ such that $t_n \rightarrow \infty$ and $x(t_n) \rightarrow z$ as $n \rightarrow \infty$. By continuity of V , it follows that $V(z) = \lambda$. Consequently,

$$V(z) = \lambda \quad \forall z \in \Omega(\xi). \quad (*)$$

By invariance of $\Omega(\xi)$, if $z \in \Omega(\xi)$, then $\varphi(t, z) \in \Omega(\xi)$ for all $t \in \mathbb{R}$ and so $V(\varphi(t, z)) = \lambda$ for all $t \in \mathbb{R}$. Therefore, $V_f(\varphi(t, z)) = 0$ for all $t \in \mathbb{R}$. Since $\varphi(0, z) = z$ and z is an arbitrary point of $\Omega(\xi)$, it follows that

$$V_f(z) = 0 \quad \forall z \in \Omega(\xi), \quad (**)$$

and so $\Omega(\xi) \subset V_f^{-1}(0)$. The claim now follows because, by Theorem 4.38, $\Omega(\xi)$ is invariant and $x(t)$ approaches $\Omega(\xi)$ as $t \rightarrow \infty$.

Comment. It might be tempting to conclude from (*) that $(\nabla V)(z) = 0$ for all $z \in \Omega(\xi)$, which then immediately would yield (**). However, this conclusion is not correct: the set $\Omega(\xi)$ is not open and therefore (*) does not imply that $(\nabla V)(z) = 0$ for all $z \in \Omega(\xi)$. (The invalidity of the conclusion is illustrated by the following simple example: if $V(z) = \|z\|^2$ and $\Omega(\xi) = \{z \in \mathbb{R}^N : \|z\| = 1\}$, then $V(z) = 1$ for all $z \in \Omega(\xi)$, but $(\nabla V)(z) = 2z \neq 0$ for all $z \in \Omega(\xi)$.)

Example.

Consider the following system of ODEs:
$$\begin{cases} x' = 2y \\ y' = -x - (1 - x^2)y \end{cases} .$$

Show the asymptotic stability of the equilibrium point in the origin and find its domain of attraction. (4p)

Solution.

We try the test function $V(x, y) = x^2 + 2y^2$ that leads to cancellation of mixed terms in the directional derivative along trajectories:

$V_f(x, y) = 4xy - 4xy - 4y^2(1 - x^2) = -4y^2(1 - x^2)$ that is not positive for $|x| \leq 1$. Therefore the origin is a stable stationary point. Checking the behavior of the system on the set of zeroes to $V_f(x, y)$ inside the stripe $|x| < 1$ we consider $(V_f)^{-1}(0) = \{(x, y) : y = 0, |x| < 1\}$. On this set $y' = -x$ and the only invariant set in $(V_f)^{-1}(0)$ is the origin. The LaSalle invariance principle implies that the origin is asymptotically stable and the domain of attraction is the largest set bounded by a level set of $V(x, y) = x^2 + 2y^2$ inside the stripe $|x| \leq 1$. The largest such set will be the interior of the ellipse $x^2 + 2y^2 = C$ such that it touches the lines $x = \pm 1$. Taking points $(\pm 1, 0)$ we conclude that $1 = C$. and the boundary of the domain of attraction is the ellipse $x^2 + 2y^2 = 1$ with halves of axes 1 and $\sqrt{0.5}$:

