Lecture notes on non-linear ODEs: limit sets (attractors), positively invariant sets, periodic solutions, limit cycles.

Plan (continuation after existence and maximal solutions)

- Semi - orbits. Limit sets. p. 142. Positively (negatively) invariant sets p. 142.
- Existence of an equilibrium point in a compact positively invariant set. Theorem 4.45, p. 150.
- Planar systems. Periodic orbits. Poincare-Bendixson theorem. (only idea of the proof is discussed) Theorem 4.46, p. 151.
- Bendixson's criterion on non existence of periodic solutions.(after lecture notes)
- First integrals and periodic orbits. Limit cycles. §4.7.2.

0.1 Introduction to limit sets and their properties.

We consider flows or dynamical systems corresponding to autonomous differential equations

$$\dot{x} = f(x), \quad f : G \to \mathbb{R}^N, \quad x(0) = \xi \tag{1}$$

with f locally Lipschitz and denote by $\varphi(t,\xi)$ the transition mapping or local flow generated by f. For $\xi \in G$ let $I_\xi = (\alpha_\xi, \omega_\xi)$ denote the maximal interval - the interval of existence of maximal solution to (1).

Definition. (Positive semi-orbit)

We denote by $O(\xi)$ the orbit of the solution to (1), $O(\xi) = \{x(t) : t \in (\alpha_\xi, \omega_\xi)\}$.

We define the positive semi-orbit $O_+(\xi) = \{x(t) : t \in [0, \omega_\xi)\}$ of ξ - for future, and negative semi-orbit (for the past) $O(\xi) = \{x(t) : t \in (\alpha_\xi, 0]\}$ o~~f~~ ξ - for the past.

Definition. (Limit point of ξ)

- A point $z \in R^N$ is called an ω - limit point of ξ (or it's positive semi-orbit $O_+(\xi)$ or it's orbit $O(\xi)$) if there is a sequence of times $\{t_n\} \in [0, \omega_\xi)$ tending to the "maximal time in the future", $t_n \nearrow \omega_\xi$ such that $\varphi(t_n, \xi) \to z$ as $n \to \infty$

• Similarly a point $z \in R^N$ is called an α - limit point of ξ (or it's negative semi-orbit $O_-(\xi)$ or it's orbit $O(\xi)$) if there is a sequence of times $\{t_n\} \in [\alpha_\xi, 0]$ tending to the "minimal time in the past", $t_n \searrow \alpha_\xi$ such that $\varphi(t_n, \xi) \to z$ as $n \to \infty$.

Definition. (ω - limit set)

The ω - limit set $\Omega(\xi)$ of ξ (or it's positive semi-orbit $O_+(\xi)$ or it's orbit $O(\xi)$) is the set of all it's ω- limit points (in future)

Definition

The α - limit set $\Omega(\xi)$ of ξ (or it's negative semi-orbit $O_-(\xi)$ or it's orbit $O(\xi)$) is the set of all it's α- limit points (in the past).

Definition. (Positively invariant set)

A set $U \subset G$ is said to be positively invariant under the local flow φ generated by f if for each starting point $\xi \in U$ from U the corresponding positive semi - orbit $O_+(\xi)$ is contained in U.

It means that all trajectories $x(t)$ starting in U stay in U as long as they exist in future.

One defines sets negatively invariant similarly, but with respect to the past.

Definition

One also says that the set U is just invariant with respect to the flowm $\varphi(t, \xi)$ if $O(\xi) \subset U$ for all $\xi \in U$. It means that all trajectories going through ξ belong to U both in the "whole past" and in the "whole future".

We know that compact positively invariant sets include trajectories that have "infinite" maximal existence time in the future: $J \cap [0, \infty)$. It makes it meaningfull to investigate limit sets of trajectories that are contained especially in compact positively invariant sets.

The first step in this kind of investigation is to identify possibly small positively invariant sets. The second step is to classify and to identify ω - limit sets that can be contained there, and that are actually contained there for a particular system.

One fundamental fact about positively invariant sets is the following.

Theorem 4.45. p. 150, L&R.

Let $C \subset G$ be non-empty, convex and compact. If C is positively invariant under the flow $\varphi(t, \xi)$, then C contains at least one equilibrium point for the corresponding ODE.

0.2 Methods of hunting positively - invariant sets

A system of ODEs has naturally many positively - invariant sets, for example the whole domain G is always a positively - invariant set, but it is not very interesting. We like to find possibly narrow invariant sets showing more precisely where trajectories or solutions to the equation tend when t tends to the upper bound of the maximal time interval.

A general idea that is used to answer many questions about behaviour of solutions (trajectories) to ODEs, is the idea of test functions. One checks if the velocities $f(x)$ are directed inside or outside with respect to the sets like $Q = \{x \in U : V(x) \leq C\}$ or $Q = \{x \in U : V(x) \geq C\}$ defined by some simple test functions $V : U \to \mathbb{R}$, $U \subset G$. A more refined variant of this idea by Lyapunov is to find test a function that is monotone along the trajectories $\varphi(t, \xi)$ of the equation. The advantage of the idea with test functions is that one does not need to solve the equation to use it.

How to find a positively - invariant set?

Method 1. We find a test function $V(x)$ that has some level sets $\partial Q = \{x : V(x) = C\}$ that are closed curves (or surfaces in higher dimensions) enclosing a bounded domain Q. Typical examples are $V(x, y) = x^2 + y^2 {=} R^2$ - circle or radius R, or $V(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} {=} 1$ - ellipse, or more complicated ones as $V(x, y) = x^6 + ay^4$ - smoothed rectangle shape or squeezed ellipse, $V(x, y) = x^2 + xy + y^2 = C$ - ellipse rotated in $\pi/4$ and having axes A and B related as $A/B = \sqrt{3}$etc.

• To show that a particular level set ∂Q bounds an positively - invariant set Q we check the sign of the directional derivative of V along the velocity in the equation $x' = f(x)$: $V_f(x) = (\nabla V \cdot f)(x)$ for all points on the level set $\{V(x) = C\}$ for a particular constant C.

• The sign of $V_f(x)$ shows if the trajectories go to the same side of the level set as the gradient ∇V (if $V(x) > 0$) or to the opposite side (if $V_f(x) < 0$).

• Then if $V(x)$ is rising for x going out of Q, and $V_f(x) < 0$ then the domain Q inside this level set ∂Q (curve in the plane case) will be positively - invariant. Similarly if $V(x)$ is decreasing out of this level set, and $V_f(x) < 0$ on the level set ∂Q then the domain Q inside this level set will be positively - invariant.

In the opposite case the complement to Q that is $\mathbb{R}^N \backslash Q$ will be positively - invariant and trajectories $\varphi(t,\xi)$ starting in this complement: $\xi \in \mathbb{R}^N \backslash Q$ will never enter Q.

First integrals. A very particular case of test functions are functions that are constant on all trajectories $\varphi(t,\xi)$ of the system. It means that $\frac{d}{dt}V(\varphi(t,\xi)) = V_f(x) \equiv 0$. Usually but not always, such test functions have the meaning of the total energy in the system. In this case all level sets of the first integral are invariant sets, because velocities $f(x)$ are tangent vectors to the level sets in this case.

Method 2. If it is difficult to guess a simple test function giving one closed formula for the boundary of an positively - invariant set as in the Method 1, then one can try to identify a boundary for an positively - invariant set as a curve (or a surface in higher dimensions) consisting of a number of simple peaces, for example straight segments.

The simplest positively - invariant set of such kind would be a rectangle (a rectangular box in higher dimensions) with sides parallel to coordinate axes. Then to check that this rectangle is an positively - invariant one needs just to check the sign of x or y - components of $f(x)$ on these segments, showing that trajectories go inside or outside of the rectangle. We had such example in the first home assignment.

Application to Poincare Bendixson theorem

One searchs often positively - invariant sets with special properties. For example to apply the Poincare-Bendixson theorem formulated later one needs to find an positively - invariant set without equilibrium points.

Example

Consider the system

$$\begin{aligned} x' &= -y + f(r)x \\ y' &= x + f(r)y \end{aligned}$$

where $r = \sqrt{x^2 + y^2}$. We will try to find an explicit expression for the corresponding flow by introducing polar coordinates $x = \cos(\theta)r$, $y = \sin(\theta)r$. We differentiate $r(t)$ using expressions for r and for x', y' in the equation, and arrive to following formulas:

$$\begin{aligned} \left(r^2\right)' &= 2rr' = (x^2 + r^2)' = 2xx' + 2yy' \\ &= 2x(-y + f(r)x) + 2y\,(x + f(r)y) = 2f(r)\left(x^2 + y^2\right) = 2f(r)r^2 \end{aligned}$$

Therefore:

$$r' = f(r)r$$

The equation for the polar angle θ can be derived by differentiating $\tan(\theta)$:

$$\begin{aligned} (\tan(\theta))' &= \frac{1}{\cos^2(\theta)}\theta' = \left(\frac{y}{x}\right)' = \frac{y'x - x'y}{x^2} \\ &= \frac{x^2 + f(r)xy - (-y^2 + f(r)xy)}{x^2} = \frac{x^2 + y^2}{x^2} = \frac{1}{\cos^2\theta} \end{aligned}$$

Therefore

$$\theta' = 1$$

The equation for $r(t)$ can be solved by integration.Each positive root r_* to $f(r)$ corrsponds to a periodic orbit $r(t) = const = r(0) = r_*$, $\theta(t) = \theta(0) + t$

This periodic orbit will attract trajectories, that start nearby if $f'(r_*) < 0$ (will be an ω - limit set $\Omega(\xi)$ for points ξ close to the circle $r = r_*$). If r_* is a root of f where the first term in Taylor series is $a(r - r_*)^2$ with $a > 0$, then nearby trajectories will be attracted to the periodic orbit from inside, and will run away from the periodic orbit from the outside of it.

Exercise 4.16

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(z) = f(z_1, z_2) := \big(z_2 + z_1(1 - \|z\|^2)\,,\, -z_1 + z_2(1 - \|z\|^2)\big).$$

Show that f generates a local flow $\varphi\colon D \to \mathbb{R}^2$ given by

$$\varphi(t, \xi) = \big(\|\xi\|^2 + (1 - \|\xi\|^2)e^{-2t}\big)^{-1/2} R(t)\xi\,,$$

where the function $R\colon \mathbb{R} \to \mathbb{R}^{2\times 2}$ is given by

$$R(t) := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \forall\, t \in \mathbb{R}. \tag{4.27}$$

(and so $R(t)\xi$ is a clockwise rotation of ξ through t radians) and

$$D := \big\{(t, \xi) \in \mathbb{R} \times \mathbb{R}^2 \colon \|\xi\|^2 + (1 - \|\xi\|^2)e^{-2t} > 0\big\}.$$

(*Hint*. Show that, for $\xi = (\xi_1, \xi_2) \neq 0$, the initial-value problem (4.25) may be expressed – in polar coordinates – as

$$\dot r(t) = r(t)\big(1 - r^2(t)\big), \quad \dot\theta(t) = -1, \quad (r(0), \theta(0)) = (r^0, \theta^0),$$

where $r^0 = \|\xi\|$, $r^0 \cos\theta^0 = \xi_1$ and $r^0 \sin\theta^0 = \xi_2$.)

Solution. The equations in polar form follow from the general argument above.

We solve the equation for r :

$$\begin{aligned} \frac{dr}{dt} &= r(1 - r^2) \\ \frac{dr}{r(1 - r^2)} &= dt \end{aligned}$$

$$\frac{1}{r(1 - r^2)} = \frac{1}{r} - \frac{1}{2\,(r + 1)} - \frac{1}{2\,(r - 1)}$$

$$
\begin{aligned}
\int \frac{dr}{r(1-r^2)} &= \ln r - \frac{1}{2}\ln\left(r^2-1\right) \\
\ln r - \frac{1}{2}\ln\left(r^2-1\right) &= t + C \\
C &= \ln|\xi| - \frac{1}{2}\ln\left(|\xi|^2-1\right) \\
\ln r - \frac{1}{2}\ln\left(r^2-1\right) - \left(\ln|\xi| - \frac{1}{2}\ln\left(|\xi|^2-1\right)\right) &= t
\end{aligned}
$$

$$
\begin{aligned}
\exp(t) &= \exp\left(\ln r - \frac{1}{2}\ln\left(r^2-1\right) - \ln|\xi| + \frac{1}{2}\ln\left(|\xi|^2-1\right)\right) \\
\frac{r}{\sqrt{r^2-1}}\frac{\sqrt{|\xi|^2-1}}{|\xi|} &= \exp(t) \\
\frac{(r^2-1)}{r^2}\frac{|\xi|^2}{\left(|\xi|^2-1\right)} &= \exp(-2t) \\
\left(r^2-1\right)|\xi|^2 &= r^2\left(|\xi|^2-1\right)\exp(-2t) \\
r^2(|\xi|^2 + \left(1-|\xi|^2\right)\exp(-2t)) &= |\xi|^2 \\
r^2 &= \frac{|\xi|^2}{(|\xi|^2 + \left(1-|\xi|^2\right)\exp(-2t))} \\
r &= \frac{|\xi|}{\sqrt{(|\xi|^2 - 1 - |\xi|^2\exp(-2t))}}
\end{aligned}
$$

Example 4.37

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be as in Exercise 4.16, the generator of the local flow

$$\varphi\colon D \to \mathbb{R}^2,\ (t,\xi) \mapsto \varphi(t,\xi) = \big(\|\xi\|^2 + (1-\|\xi\|^2)e^{-2t}\big)^{-1/2} R(t)\xi\,,$$

where $R : \mathbb{R} \to \mathbb{R}^{2\times 2}$ is defined by (4.27). The domain D of φ is

$$D = \big\{(t,\xi) \in \mathbb{R}\times\mathbb{R}^2 \colon \|\xi\|^2 + (1-\|\xi\|^2)e^{-2t} > 0\big\}.$$

Let Δ be the open unit disc in $\mathbb{R}^2$, that is, $\Delta = \{(z_1,z_2) \in \mathbb{R}^2 \colon z_1^2 + z_2^2 < 1\}$.

The aims of this example are (i) to show that the sets $\{0\}$, Δ, $\partial\Delta$ and $\mathbb{R}^2\backslash\bar{\Delta}$ are invariant, and (ii) to find, for every $\xi \in \mathbb{R}^2$, the associated ω and α-limit set.

Since, for each $t \in \mathbb{R}$, $R(t)$ is an orthogonal matrix (representing a rotation through t radians), we have that $\|R(t)\xi\| = \|\xi\|$ for each $\xi \in \mathbb{R}^2$, a fact which will be used freely in the arguments below.

Case 1. If $\xi = 0$, then $\varphi(t,\xi) = 0$ for all $t \in I_\xi = \mathbb{R}$ and thus the set $\{0\}$ is invariant and

$$A(\xi) = O(\xi) = \Omega(\xi) = \{0\}.$$

Case 2. If $\|\xi\| = 1$, then, for all $t \in I_\xi = \mathbb{R}$, $\varphi(t,\xi) = R(t)\xi$ and thus, in particular, $\|\varphi(t,\xi)\| = 1$. We conclude that $\partial\Delta$ is invariant and

$$A(\xi) = O(\xi) = \Omega(\xi) = \partial\Delta.$$

Case 3. Assume $0 < \|\xi\| < 1$. Then $\|\varphi(t,\xi)\| < 1$ for all $t \in I_\xi = \mathbb{R}$, showing that Δ is invariant. Furthermore, let $z \in \partial\Delta$ be arbitrary and let $T \geq 0$ be such that $R(T)\xi = \|\xi\|z$. Define the sequence (t_n) by $t_n := T + 2n\pi$. Then

$$\varphi(t_n,\xi) = \big(\|\xi\|^2 + (1-\|\xi\|^2)e^{-2t_n}\big)^{-1/2} R(T)\xi \to z \ \text{ as } n \to \infty\,,$$

and so $z \in \Omega(\xi)$. Since $z \in \partial\Delta$ is arbitrary, we have $\partial\Delta \subset \Omega(\xi)$. Moreover, $\|\varphi(t,\xi)\| \to 1$ as $t \to \infty$, and thus it is clear that there are no other ω-limit points, showing that $\Omega(\xi) = \partial\Delta$. Since $\|\varphi(t,\xi)\| \to 0$ as $t \to -\infty$, it follows that $A(\xi) = \{0\}$.

Case 4. Finally, consider the remaining case wherein $\|\xi\| > 1$. Here (the only case in which $I_\xi \neq \mathbb{R}$), the maximal interval of existence I_ξ is given by $I_\xi = (\alpha_\xi\,,\,\infty)$, where

$$\alpha_\xi := \ln\left(\sqrt{\|\xi\|^2 - 1}/\|\xi\|\right) < 0.$$

Since $\|\varphi(t,\xi)\| > 1$ for all $t \in I_\xi$, we see that $\mathbb{R}^2\backslash\bar{\Delta}$ is invariant. Moreover, by the same argument as used in Case 3, we find that $\Omega(\xi) = \partial\Delta$. Finally, $\|\varphi(t_n,\xi)\| \to \infty$ as $t \to \alpha_\xi$, and so we may conclude that $A(\xi) = \emptyset$. $\triangle$

Calculation of α_ξ is given here:

$$\begin{aligned}
\|\xi\|^2 + \left(1 - \|\xi\|^2\right) e^{-2t} &= 0 \\
\frac{\|\xi\|^2}{\|\xi\|^2 - 1} &= e^{-2t} \\
&= e^{t} \\
\ln\left(\frac{\sqrt{\left(\|\xi\|^2 - 1\right)}}{\|\xi\|}\right) &= t = \alpha_\xi < 0
\end{aligned}$$

0.3 Dynamical systems in plane. Poincare Bendixson theorem, periodic solutions and more positively invariant sets..

Theorem. Poincare-Bendixson theorem.

if $\xi \in G$ is such that the closure of the positive orbit $O_+(\xi)$ is compact and is contained in G and the ω limit set $\Omega(\xi)$ does not contain equilibrium points, then the ω - limit set $\Omega(\xi)$ is an orbit of a periodic solution.

Definition

A periodic orbit γ (an orbit corresponding to a periodic solution) is called an ω - limit cycle (or often just a limit cycle) if $\gamma = \Omega(\xi)$ for some $\xi \in G\backslash\gamma$: γ is an ω-limit set for some point ξ outside γ.

This definition excludes the case of phase portraits completely filled periodic orbits, as the system $x' = -y$, $y' = x$, having all orbits being circles arund the origin.

Hint to applications. It is easier to check that there is a compact positively invariant set $C \subset G \subset \mathbb{R}^2$ such that $\xi \in C$. If C contains no equilibrium points, then the ω - limit set $\Omega(\xi)$ of ξ is an orbit of a periodic solution.

On the other hand Theorem 4.45 suggests that any periodic orbit in plane encloses a compact positively invariant set that includes at least one equilibrium point. It means that a typical compact positively - invariant set for applying the Poincare-Bendixson theorem should be a closed ring shaped set with at least one hole in the middle including a repelling non stable equilibrium point.

Check list for application of the Poincare-Bendixson theorem.

• One starts with applying one of the two methods above to find a compact positively - invariant set Q with at least one equilibrium point inside it. Such set Q does not satisfy conditions in the Poincare-Bendixson theorem yet.

• To identify holes around the equilibriums in the middle (one must find all such equilibrium points at the end !), one needs often to find one more test function for each of them, to show that trajectories do not enter a neighbourhood of each of the equilibriums.

• Alternatively one can use the linearization to show that this equilibrium is repeller and therefore trajectories cannot enter some small neighbourhood of the equilibrium in the middle of the set Q.

• One must check at the end that the positively invariant annulus (closed ring shaped domain) does not include equilibrium points (no at the boundary either!). It is often simpler to do it after carrying out estimates for V_f by first checking zeroes of $V_f(x) = 0$ that contain naturally all equilibrium points but is a scalar equation, and then checking zeroes of the system $f(x) = 0$.

Example. Show that the system has a periodic solution.

Solution. First, we convert Equation (7) to a system of two first-order equations by setting $x = z$ and $y = \dot{z}$. Then,

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x + (1 - x^2 - 2y^2)y. \tag{8}$$

Next, we try and find a bounded region R in the $x-y$ plane, containing no equilibrium points of (8), and having the property that every solution $x(t)$, $y(t)$ of (8) which starts in R at time $t = t_0$, remains there for all future time $t \geqslant t_0$. It can be shown that a simply connected region such as a square or disc will never work. Therefore, we try and take R to be an annulus surrounding the origin. To this end, compute

$$\frac{d}{dt}\left(\frac{x^2+y^2}{2}\right) = x\frac{dx}{dt} + y\frac{dy}{dt} = (1 - x^2 - 2y^2)y^2,$$

and observe that $1 - x^2 - 2y^2$ is positive for $x^2 + y^2 < \frac{1}{2}$ and negative for $x^2 + y^2 > 1$. Hence, $x^2(t) + y^2(t)$ is increasing along any solution $x(t)$, $y(t)$ of (8) when $x^2 + y^2 < \frac{1}{2}$ and decreasing when $x^2 + y^2 > 1$. This implies that any solution $x(t)$, $y(t)$ of (8) which starts in the annulus $\frac{1}{2} < x^2 + y^2 < 1$ at time $t = t_0$ will remain in this annulus for all future time $t \geqslant t_0$. Now, this annulus contains no equilibrium points of (8). Consequently, by the Poincaré–Bendixson Theorem, there exists at least one periodic solution $x(t)$, $y(t)$ of (8) lying entirely in this annulus, and then $z = x(t)$ is a nontrivial periodic solution of (7).

Example. Arrowsmith-Place. One can instead of the analytical approach shown below, use a more geometric argument, based on considering the curves const $= 3x_1^2 + 2x_2^2$. It is demonstrated later for the **Exercise 4.21.**

Example 3.9.1. Show that the phase portrait of

$$\ddot{x} - \dot{x}(1 - 3x^2 - 2\dot{x}^2) + x = 0$$

has a limit cycle.

Solution. The corresponding first-order system is

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2), \tag{3.74}$$

which becomes

$$\dot{r} = r\sin^2\theta(1 - 3r^2\cos^2\theta - 2r^2\sin^2\theta) \tag{3.75}$$

$$\dot{\theta} = -1 + \tfrac{1}{2}\sin 2\theta(1 - 3r^2\cos^2\theta - 2r^2\sin^2\theta) \tag{3.76}$$

in polar coordinates. Observe:

(a) Equation (3.75) with $r = \frac{1}{2}$ gives

$$\dot{r} = \tfrac{1}{4}\sin^2\theta(1 - \tfrac{1}{2}\cos^2\theta) \geqslant 0 \tag{3.77}$$

with equality only at $\theta = 0$ and π. Thus, $\{\mathbf{x} \mid r > \frac{1}{2}\}$ is positively invariant;

(b) Equation (3.75) also implies that

$$\dot{r} \leqslant r\sin^2\theta(1 - 2r^2).$$

Thus for $r = 1/\sqrt{2}$, $\dot{r} \leqslant 0$ with equality only at $\theta = 0$ and π. Thus, $\{\mathbf{x} \mid r < 1/\sqrt{2}\}$ is positively invariant.

Now (a) and (b) imply that the annular region $\{\mathbf{x} \mid \frac{1}{2} < r < 1/\sqrt{2}\}$ is positively invariant and, since the only fixed point of (3.74) is at the origin, we conclude there is a limit cycle in the annulus $\square$

As Theorem 4.45 and examples considered before suggests the positively invariant set we look for must have a shape of annulus with a hole in the middle containing at least one equilibrium point. The next Proposition 4.56, p. 165 gives a particular hint how to find the "hole" for such an annulus domain using the techniques of stability by linearization (Grobman-Hartman theorem) that we studied earlier.

Proposition 4.56

Let $C \subset G$ be a compact set that is positively invariant under the local flow φ generated by (4.39). Assume that 0 is an interior point of C and is the only equilibrium in C. Assume further that f is differentiable at 0. Write $A := (Df)(0)$ with spectrum $\sigma(A) = \{\lambda_1, \lambda_2\}$. If $\operatorname{Re} \lambda_i > 0$ for $i = 1, 2$, then there exists at least one ω-limit cycle in C.

Proof

By positive invariance of C, the positive semi-orbit $O^+(\xi)$ of every $\xi \in C$ is contained in C. Therefore, the closure of $O^+(\xi)$ is a compact subset of G and so $\mathbb{R}_+ \subset I_\xi$ for every $\xi \in C$. The hypotheses, together with Theorem 5.33 (a result to be established in the next chapter), ensure that the equilibrium 0 is repelling in the sense it has a neighbourhood U such that U is contained in C and, for each $\xi \in U \backslash \{0\}$, there exists $\tau \in \mathbb{R}_+$ such that $\varphi(t, \xi) \notin U$ for all $t \geq \tau$. Let $\xi \in U$ be arbitrary. Then $\Omega(\xi) \subset C \backslash U$ and so contains no equilibrium points. By the Poincaré-Bendixson theorem (Theorem 4.46), it follows that $\Omega(\xi)$ is the orbit of a periodic point. Finally, since ξ is in U, we have $O^+(\xi) \neq \Omega(\xi)$ and so we may conclude that $\Omega(\xi)$ is an ω-limit cycle. $\square$

Example 4.57

Consider again the system given in Exercise 4.16, with $G = \mathbb{R}^2$ and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(z) = f(z_1, z_2) := \big(z_2 + z_1(1 - \|z\|^2)\,,\, -z_1 + z_2(1 - \|z\|^2)\big)$. Let C be the closed unit disc $\{z \in \mathbb{R}^2 \colon \|z\| \leq 1\}$. Then

$$\langle z, f(z)\rangle = \|z\|^2\big(1 - \|z\|^2\big) = 0 \quad \forall z \in \partial C,$$

and so solutions starting in C cannot exit C in forwards time. Thus, the compact set C is positively invariant. Moreover, 0 is the unique equilibrium in C

and

$$A = (Df)(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

with spectrum $\sigma(A) = \{1 + i, 1 - i\}$. Therefore, by Proposition 4.56, we may conclude the existence of a limit cycle in C. This, of course, is entirely consistent with Exercise 4.16 and Example 4.37, the conjunction of which shows (by explicit computation of the local flow) that the unit circle $\gamma = \partial C$ is a periodic orbit and coincides with the ω-limit set $\Omega(\xi)$ of every ξ with $0 < \|\xi\| < 1$. $\triangle$

Exercise 4.21, p. 158

Consider the system $z' = f(z_1, z_2)$:

$$\begin{aligned} z_1' &= z_2 + z_1 g(z_1, z_2) \\ z_2' &= -z_1 + z_2 g(z_1, z_2) \\ g(z_1, z_2) &= 3 + 2z_1 - z_1^2 - z_2^2 \end{aligned}$$

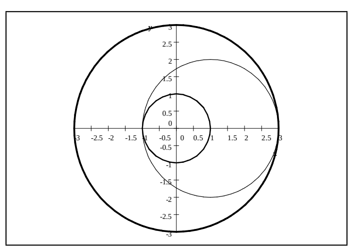
Prove that the system has at least one periodic solution.

Solution.

Consider the test function $V(z_1, z_2) = \left(\frac{z_1^2+z_2^2}{2}\right)$.

$$\begin{aligned} \nabla V \cdot f(z_1, z_2) &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \cdot \begin{bmatrix} z_2 + z_1 g(z_1, z_2) \\ -z_1 + z_2 g(z_1, z_2) \end{bmatrix} \\ &= \left(z_1^2 + z_2^2\right) g(z_1, z_2) = \left(z_1^2 + z_2^2\right) \left(3 + 2z_1 - z_1^2 - z_2^2\right) \\ &= r^2(4 - (1 - z_1)^2 - z_2^2) \end{aligned}$$

The circle $4 = (1 - z_1)^2 + z_2^2$ has center in the point $(1, 0)$ and radius 2:



Inside this circle $\nabla V \cdot f(z_1, z_2) > 0$ outside this circle $\nabla V \cdot f(z_1, z_2) < 0$. Therefore as it is easy to see from the picture, $\nabla V \cdot f(z_1, z_2) \geq 0$ on the circle $z_1^2 + z_2^2 = 1$ with center in the origin, and $\nabla V \cdot f(z_1, z_2) \leq 0$ on the circle $z_1^2 + z_2^2 = 9$ with center in the origin. The ring shaped set C: $1 \leq r \leq 3$ is a positively invariant compact set. The origin is the only equilibrium point fo the system, because from the expression $\nabla V \cdot f(z_1, z_2) = r^2 g(z_1, z_2)$ it follows that other equilibrium points must be on the circle $g(z_1, z_2) = 0 = 4 - (1 - z_1)^2 - z_2^2$. Substitution $g(z_1, z_2) = 0$ into the system leads to the conclusion that there are no equilibrium points on this circle.

Therefore the Poincare Bendixson theorem implies that there exists at least one periodic orbit contained in the ring shaped set C.

Exercise.

Solve a similar problem for the function $g(z_1, z_2) = 3 + z_1 z_2 - z_1^2 - z_2^2$

Example 3.8.2. Find the limit cycles in the following systems and give their types:

(a) $\dot{r} = r(r-1)(r-2), \quad \dot{\theta} = 1;$ (3.67)

(b) $\dot{r} = r(r-1)^2, \quad \dot{\theta} = 1.$ (3.68)

Solution

(a) There are closed trajectories given by

$$r(t) \equiv 1, \quad \theta = t \quad \text{and} \quad r(t) \equiv 2, \quad \theta = t. \tag{3.69}$$

Furthermore

$$\dot{r} \begin{cases} > 0, & 0 < r < 1 \\ < 0, & 1 < r < 2 \\ > 0, & r > 2 \end{cases} . \tag{3.70}$$

The system therefore has two circular limit cycles: one stable ($r = 1$) and one unstable ($r = 2$).

(b) System (3.68) has a single circular limit cycle of radius one. However, $\dot{r}$ is positive for $0 < r < 1$ *and* $r > 1$, so the limit cycle is semistable. □

Counterexample: an annulus that is a region of attraction containing an attracting equilibrium.

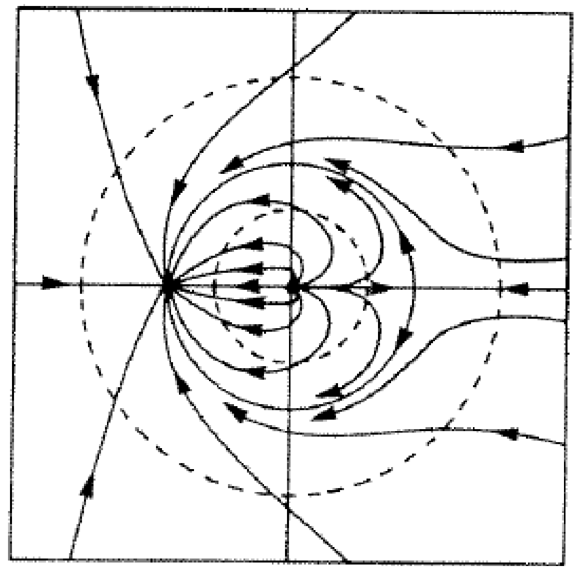


Fig. 3.25. Phase portrait for the system $\dot{r} = r(1-r)$, $\dot{\theta} = \sin\theta$.

Exercise. Rectangular positively invariant set

Consider the following system of ODEs :

$$\begin{cases} x' = & 10 - x - \frac{4xy}{1+x^2} \\ y' = & x\left(1 - \frac{y}{1+x^2}\right) \end{cases}$$

1. a) show that the point (x_*, y_*) with coordinates $x_* = 2$ and $y_* = 5$ is the only equilibrium point and is a repeller;

   b) find a rectangle $[0,a]\times[0,b]$ in the first quadrant $x > 0$, $y > 0$ bounded by coordinate axes and by two lines parallel to them, that is a positively invariant set. Explain why the system must have at least one periodic orbit in this rectangle.

   Solution.

   a) $x_* = 2$ and $y_* = 5$ is an equilibrium point:$\left(1 - \frac{5}{1+2^2}\right) = 0$; and $10 - 2 - \frac{4\cdot 2\cdot 5}{5} = 10 - 2 - 8 = 0$.

   The Jacobi matrix is $A = \begin{bmatrix} -4\frac{y}{x^2+1} + 8x^2\frac{y}{(x^2+1)^2} - 1 & -4\frac{x}{x^2+1} \\ -\frac{y}{x^2+1} + 2x^2\frac{y}{(x^2+1)^2} + 1 & -\frac{x}{x^2+1} \end{bmatrix}$. It is calculated as:

   $$\nabla\left(10 - x - \frac{4xy}{1+x^2}\right) = \left.\begin{bmatrix} -4\frac{y}{x^2+1} + 8x^2\frac{y}{(x^2+1)^2} - 1 \\ -4\frac{x}{x^2+1} \end{bmatrix}\right|_{x=2,\ y=5}$$

   $$= \begin{bmatrix} -4\frac{5}{5} + 8\,(4)\,\frac{5}{25} - 1 \\ -4 * \frac{2}{5} \end{bmatrix} = \begin{bmatrix} -4 + \frac{32}{5} - 1 \\ -\frac{8}{5} \end{bmatrix} = \begin{bmatrix} 1.4 \\ -1.6 \end{bmatrix}$$

   $$\nabla\left(x\left(1 - \frac{y}{1+x^2}\right)\right) = \left.\begin{bmatrix} -\frac{y}{x^2+1} + 2x^2\frac{y}{(x^2+1)^2} + 1 \\ -\frac{x}{x^2+1} \end{bmatrix}\right|_{x=2,\ y=5} = \begin{bmatrix} -\frac{5}{5} + 2(4)\frac{5}{25} + 1 \\ -\frac{2}{5} \end{bmatrix}$$

   $$= \begin{bmatrix} -1 + \frac{8}{5} + 1 \\ -\frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1.6 \\ -0.4 \end{bmatrix}$$

   The Jacobi matrix in x_*, y_* is $A = \begin{bmatrix} 1.4 & -1.6 \\ 1.6 & -0.4 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - \lambda + 2 = 0$,

   $trace(A) = 1 > 0$, $\det(A) = 2 > \frac{[trace(A)]^2}{4} = \frac{1}{4}$ that corresponds to unstable spiral and it is a repeller, eigenvalues are: $\lambda_1 = 0.5 + \sqrt{0.25 - 2} = 0.5 + i\sqrt{1.75}$, $\lambda_2 =$

$0.5 - \sqrt{0.25 - 2} = 0.5 - i\sqrt{1.75}$.

It implies that trajectories cannot enter a small open ball $B((x_*, y_*), \varepsilon)$ with the center in the equilibrium point $(2, 5)$ and some small radius ε.

b) Observe that the first quadrant is a positively invariant set. For $y = 0$ we have $\dot{x} = 10 > 0$ and for $y = 0$ and $x > 0$ we have $y' = x > 0$.

Observe also that $\dot{y} < 0$ for $y > 1 + x^2$ and $x > 0$; $\dot{x} < 0$ for $x > 10$ and $y > 0$.

It implies that the rectangle $[0, 10] \times [0, 101]$ is a positively invariant compact set. Excluding a small open ball $B((x_*, y_*), \varepsilon)$ with the center in the equilibrium point $(2, 5)$ and small radius ε we get a positively invariant compact set $[0, 10] \times [0, 101] \setminus B((x_*, y_*), \varepsilon)$ without equilibrium points that according to the Poincare Bendixson theorem must include a periodic orbit.

The following theorem gives a more complete description of the types of ω - limit sets in the plane $\mathbb{R}^2$.

Theorem 7.16 (generalized Poincaré–Bendixson). *Let M be an open subset of $\mathbb{R}^2$ and $f \in C^1(M, \mathbb{R}^2)$. Fix $x \in M$, $\sigma \in \{\pm\}$, and suppose $\omega_\sigma(x) \neq \emptyset$ is compact, connected, and contains only finitely many fixed points. Then one of the following cases holds:*

(i) *$\omega_\sigma(x)$ is a fixed orbit.*

(ii) *$\omega_\sigma(x)$ is a regular periodic orbit.*

(iii) *$\omega_\sigma(x)$ consists of (finitely many) fixed points $\{x_j\}$ and non-closed orbits $\gamma(y)$ such that $\omega_\pm(y) \in \{x_j\}$.*

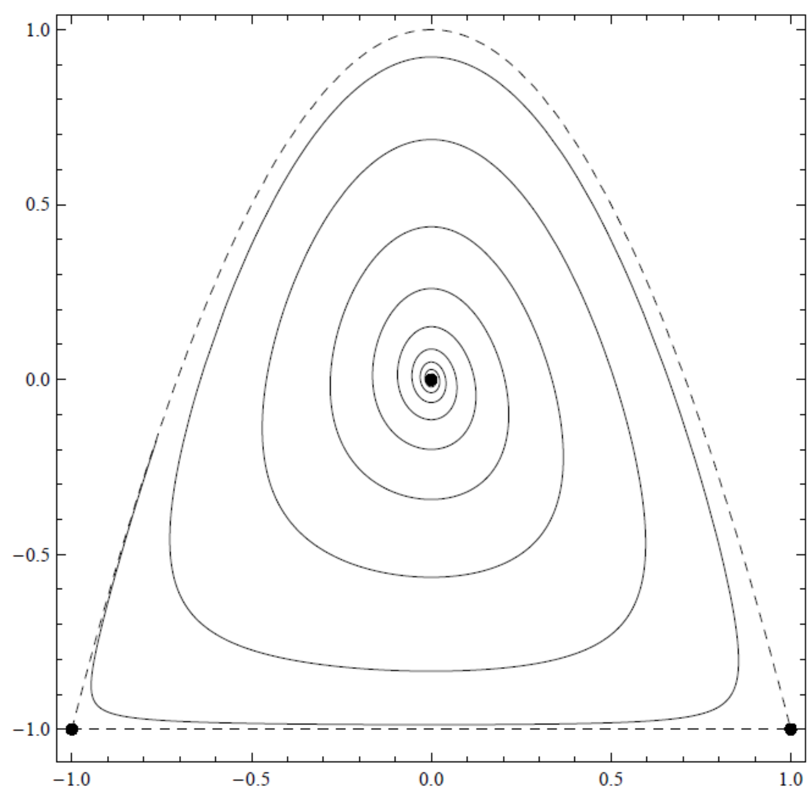
Proof. If $\omega_\sigma(x)$ contains no fixed points it is a regular periodic orbit by Lemma 7.13. If $\omega_\sigma(x)$ contains at least one fixed point x_1 but no regular points, we have $\omega_\sigma(x) = \{x_1\}$ since fixed points are isolated and $\omega_\sigma(x)$ is connected.

Suppose that $\omega_\sigma(x)$ contains both fixed and regular points. Let $y \in \omega_\sigma(x)$ be regular. We need to show that $\omega_\pm(y)$ consists of one fixed point. Therefore it suffices to show that it cannot contain regular points. Let $z \in \omega_\pm(y)$ be regular. Take a transversal arc Σ containing z and a sequence $y_n \to z$, $y_n \in \gamma(y) \cap \Sigma$. By Corollary 7.10 $\gamma(y) \subseteq \omega_\sigma(x)$ can intersect Σ only in y. Hence $y_n = z$ and $\gamma(y)$ is regular periodic. Now Lemma 7.14 implies $\gamma(y) = \omega_\sigma(x)$ which is impossible since $\omega_\sigma(x)$ contains fixed points. □

Example. While we have already seen examples for case (i) and (ii) in the Poincaré–Bendixson theorem we have not seen an example for case (iii). Hence we consider the vector field

$$f(x,y) = \begin{pmatrix} y + x^2 - \alpha x(y - 1 + 2x^2) \\ -2(1+y)x \end{pmatrix}.$$

First of all it is easy to check that the curves $y = 1 - 2x^2$ and $y = -1$ are invariant. Moreover, there are four fixed points $(0,0)$, $(-1,-1)$, $(1,-1)$, and $(\frac{1}{2\alpha}, -1)$. We will chose $\alpha = \frac{1}{4}$ such that the last one is outside the region bounded by the two invariant curves. Then a typical orbit starting inside this region is depicted in Figure 7.9: It converges to the unstable fixed point $(0,0)$ as $t \to -\infty$ and spirals towards the boundary as $t \to +\infty$. In



particular, its $\omega_+((x_0, y_0))$ limit set consists of three fixed points plus the orbits joining them.

To prove this consider $H(x, y) = x^2(1 + y) + \frac{y^2}{2}$ and observe that its change along trajectories

$$\dot{H} = 2\alpha(1 - y - 2x^2)x^2(1 + y)$$

is nonnegative inside our region (its boundary is given by $H(x, y) = \frac{1}{2}$). Hence it is straightforward to show that every orbit other than the fixed point $(0, 0)$ converges to the boundary. ◇

0.4 Methods of hunting ω - limit sets

0.5 How to find an ω - limit set?

We put here this user guide about ω **- limit sets** that refers to some notions that will be discussed later in the course. You can come back to this text when corresponding notions will be introduced.

ω - limit sets live naturally inside ω - invariant sets. In case one can find a very small ω - invarinat set the position and the size of the ω - limit set inside it will be rather well defined.

Description properties of ω - limit sets is the main and the most complicated problem in the theory of dynamical systems. Even numerical investigation of limit sets in dimension higher then 2 is rather complicated and needs advanced mathematical tools.

In autonomous systems the plane $\mathbb{R}^2$ limit sets can be only of three types: a) **equilibrium points, b) periodic orbits, and c) closed curves consisting of finite number of equilibrium points connected by open orbits.** It is an extension of the Poincare-Bendixson theorem.

The analytic identification or at least effective localization of ω - limit sets is possible with help of La Salle's invariance theorem that will be studied later. It states that ω - limit sets are subsets of zero level sets of $V_f(x) = (\nabla V \cdot f)(x)$ for appropriate test function (Lyapunov function) $V(x)$ satisfying $V_f(x) \leq 0$.

This theorem helps in particular to find ω - limit sets that are asymptotically stable equilibrium points, by a rather simple check of the behaviour of the velocity $f(x)$ on the zero level set where $V_f(x) = 0$.

One can also investigate asymptotically stable equilibrium points with help of so called "strong" Lyapunov functions V that satisfy $V_f(x) < 0$ for $x \neq 0$.

It is difficult in practice to find analytically ω - limit sets in plane of two other types. It is possible if one can find analytically a zero level set $V_f^{-1}(0)$ of a test function V that is a closed curve in plane. Then this level set belongs to one of the two other types: periodic orbit or a chain of equilibrium points connected by open orbits.

Such an analytic construction is not known for the equation with periodic orbit in the second home assignment, despite the fact that special techniques were developed to show that the periodic orbit there is unique.

If a system has first integrals: test functions having $V_f(x) = 0$ everywhere, then level sets of first integrals give a good tool to identify ω - limit sets because these level sets consist of orbits and are because of that very narrow invariant sets. The existence of first integrals is usually a sign that the energy of the system is preserved, that is a rather special situation.

The observations above show that in many practical situations we can find ω - limit sets that are asymptotically stable equilibrium points.

For systems in plane we can with help of Poincare Bendixson theorem also show that in certain situations ω - limit sets are orbits of periodic solutions but cannot give a formula for them and cannot state how many they are.

ω - limit sets in the plane that are more complicated than equilibrium points, is possible to describe analytically in the case when for a Lyapunov test function $V(x)$ the zero level set $V_f^{-1}(0)$ is a closed curve in the plane and the corresponding equation can be investigated analytically.