3.5 Linear Systems with Periodic Coefficients

In this section, we shall study the linear periodic systems

$$x' = A(t)x, \quad A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}, \tag{LP}$$

where $A(t)$ is continuous on $\mathbb{R}$ and is periodic with period T, i.e., $A(t) = A(t+T)$ for all t. We shall analyze the structure of the solutions $x(t)$ of (LP). Before we prove the main results we need the following theorem concerning the logarithm of a nonsingular matrix.

Theorem 3.5.1 *Let $B \in \mathbb{R}^{n\times n}$ be nonsingular. Then there exists $A \in \mathbb{C}^{n\times n}$, called logarithm of B, satisfying $e^A = B$.*

Proof. Let $B = PJP^{-1}$ where J is a Jordan form of B, $J =$ diag

$(J_0, J_1, \cdots, J_s)$ with

$$J_0 = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & 0 \\ 0 & & \ddots & \\ & & & \lambda_k \end{bmatrix} \text{ and } J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i},$$

$i = 1, \cdots, s.$

Since B is nonsingular, $\lambda_i \neq 0$ for all i. If $J = e^{\tilde{A}}$ for some $\tilde{A} \in \mathbb{C}^{n \times n}$ then it follows that $B = Pe^{\tilde{A}}P^{-1} = e^{P\tilde{A}P^{-1}} \stackrel{\text{def}}{=} e^{A}$. Hence it suffices to show that the theorem is true for Jordan blocks J_i, $i = 1, \cdots, s$. Write

$$J_i = \lambda_i \left(I + \frac{1}{\lambda_i} N_i \right), \ N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

Then $N_i^{n_i} = O$. From the identity

$$\log(1 + x) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} x^p, \ |x| < 1$$

and

$$e^{\log(1+x)} = 1 + x, \tag{3.11}$$

we formally write

$$\begin{aligned} \log J_i &= (\log \lambda_i) I + \log \left(I + \tfrac{1}{\lambda_i} N_i \right) \\ &= (\log \lambda_i) I + \textstyle\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \left(\frac{N_i}{\lambda_i} \right)^p. \end{aligned} \tag{3.12}$$

From (3.12) we define

$$A_i = (\log \lambda_i) I + \sum_{p=1}^{n_i - 1} \frac{(-1)^{p+1}}{p} \left(\frac{N_i}{\lambda_i} \right)^p.$$

Then from (3.11) we have

$$e^{A_i} = \exp((\log \lambda_i) I) \exp \left(\sum_{p=1}^{n_i - 1} \frac{(-1)^{p+1}}{p} \left(\frac{N_i}{\lambda_i} \right)^p \right) = \lambda_i \left(I + \frac{N_i}{\lambda_i} \right) = J_i.$$

□