1. **Lyapunov function and stability of stationary points.**

Consider the system of ODE: $\begin{array}{l} x' = y \\ y' = -y + y^3 - x^5 \end{array}$

Write the definition of asymptotically stable stationary point.

Find Lyapunov function $V(x,y)$ for the given equation and show that stationary point in the origin is asymptotically stable. **(2p)**

Hint. Use $V(x,y)$ in the form form $V(x,y) = ax^6 + by^2$ and choose the parameters a, b so that $V(x,y)$ would be a Lyapunov function.

Solution A stationary point x_* is asymptotically stable if there is a neighborhood N of this point such that for any initial point $x(o) \in N$ the corresponding trajectory $x(t) \to x_*$ for $t \to +\infty$.

$V'(x,y) = \frac{\partial V}{\partial x} y + \frac{\partial V}{\partial x}(-y + y^3 - x^5) = 6ax^5 y - 2by^2 + 2by^4 - 2bx^5 y$

Taking $6a = 2b$ and $a = 1$ makes $V'(x,y) = 6y^2(1 - y^2) < 0$ for $|y| < 1$ and $y \neq 0$.

For $y = 0$ $y' = -x^5$ and therefore $y' = 0$ only for $x = 0$. It implies that the origin iz asymptotically stable.

1. **Lyapunovs funktioner och stabilitet hos stationära punkter.**

a) Formulera ett criterium for asymptotiskt stabil stationär punkt till en ODE med hjälp av en Lyapunovs funktion som inte är stark Lyapunovs funktion. **(2p)**

b) Betrakta ekvationssystemet: $\begin{array}{l} x' = -y/3 - x(3x^2 + y^2) \\ y' = x - y(3x^2 + y^2) \end{array}$

Hitta en stark Ljapunovs funktion $V(x,y)$ för att visa att stationära punkten i origo är asymptotiskt stabil. **(2p)**

Tips. Använd $V(x,y)$ på formen $V(x,y) = ax^2 + y^2$ och välj parametern a så att $V(x,y)$ blir en stark Ljapunovs funktion.

$V(x,y)$ kan väljas som $V(x,y) = 3x^2 + y^2$.

2. **Lyapunovs functions and stability of stationary points.** Formulate the criterion for asymptotic stability of a stationary point of an ODE using only a weak Lyapunov function.

Consider the system of equations: $\begin{cases} x' = -x + y^2 \\ y' = -xy - x^2 \end{cases}$

Show that $V(x,y) = x^2 + y^2$ is a weak Lyapunov function and decide if the stationary point at the origin is asymptotically stable. **(4p)**

$\frac{d}{dt}(V) = \nabla V \cdot \begin{bmatrix} -x + y^2 \\ -xy - x^2 \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -x + y^2 \\ -xy - x^2 \end{bmatrix} = 2x\left(-x + y^2\right) + 2y\left(-xy - x^2\right) =$
$-2x^2 - 2x^2 y = -2x^2(1+y)$;

For $|y| < 1$ $\frac{d}{dt}(V) \leq 0$. It implies that the origin is a stable stationary point. On the line $x = 0$ $\frac{d}{dt}(V) = 0$ so V is only a weak Lyapunovs funktion. But we observe that on the line $x = 0$ the velocity is not zero $-x + y^2 \neq 0$ exept the stationary point itself. It implies that the origin is an asymptoticaly stable stationary point.

2. **Ljapunovs functions and stability of stationary points.**

Consider the system of equations: $\begin{cases} x' = -x + 2xy^2 \\ y' = -(1-x^2)y^3 \end{cases}$

Show that the origin is an stable fixed point. **(4p)**

Solution

$V(x,y) = x^2 + y^2$

$V' = 2x\left(-x + 2xy^2\right) + 2y\left(-(1-x^2)y^3\right) = -2x^2 + 4x^2y^2 - 2y^4\left(1 - x^2\right) =$
$-2x^2\left(1 - 2y^2\right) - 2y^4\left(1 - x^2\right)$

We see that $V' < 0$ for $|x| < 1$ and $|y| < \sqrt{1/2}$

Exercise.

Consider the system of equations

$$\begin{aligned} x' &= -x + 2xy\sin(y) \\ y' &= -\cos(x)y \end{aligned}$$

Investigate stability of the fixed point in the origin.

Linearization gives Jacoby matrix:

$A(x,y) = \begin{bmatrix} -1+2y\sin(y) & 2x(\sin(y)+y\cos(y)) \\ \sin(x)y & -\cos(x) \end{bmatrix}$

$A(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. It implies that the equilibrium is asymptotically stable.

We try also using a Lyapunov function. But it feels an overkill comparing with linearization!

But with hard additional work estimating constants in errors of linearizations we could specify a region of attraction. We will not do it here.

We try $V(x,y) = x^2 + y^2$ and use the Cauchy inequality $2ab \le a^2 + b^2$ and linearization for $\sin(y)$ and $\cos(x)$ when $y \to 0$ and $x \to 0$: $\sin(y) = y + O(y^2)$ and $(1-\cos(x)) = x^2 + O(x^3)$. The notation $O(x)$ means that $O(x)/x$ is bounded when $x \to 0$.

$V_f(x,y) = 2x\left(-x + 2xy\sin(y)\right) + 2y\left(-\cos(x)y\right) = 4x^2y\sin y - 2y^2\cos x - 2x^2 =$

$4x^2y\sin y + 2y^2\left(1-\cos x\right) - 2y^2 - 2x^2 = 4x^2y(y+O(y^2)) + 2y^2\left(O(x^2)\right) - 2y^2 - 2x^2 =$

$-2y^2 - 2x^2 + 4x^2y^2 + x^2O(y^3) + 2y^2O(x^2) \le -2\left(x^2+y^2\right) + 2(x^4+y^4) + x^2O(y^3) + 2y^2O(x^2)$

For small neighbourhood N of the origin the first term $-2\left(x^2+y^2\right)$ dominates all other terms of higher order.

It implies that in $V_f(x,y) < 0$ for $(x,y) \ne (0,0)$ in N and the origin is asymptotically stable.