MATEMATIK	Datum: 2015-08-24	Tid: 14:00
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Lösningar till tenta i ODE och matematisk modellering, MMG511, MVE161

Answer first those questions that look simpler, then take more complicated ones etc.
Good luck!

1. Formulate general properties of solutions to linear (non-autonomous) systems of ODE.

   Give a proof to the statement about the dimension of the space of all solutions to a linear system of ODE. **(4p)**

   Answer.

   i) A linear combination of solutions to a linear system of ODE's is the solutuion to the same system and the set of all solutions is a linear vector space.

   ii) Linear systems of equations have a unique solution to any initial value problem to the system

   iii) The set of all solutions to a linear system of ODE's $x' = A(t)x$ with matrix $A(t)$ of size $n \times n$ is a linear space of dimension n.

   To prove the last statement we take n linearly independent vectors $v_k \in \mathbb{R}^n$, $k = 1...n$.

   Any vector L_0 in $\mathbb{R}^n$ can be represented as a linear combination $L_0 = \sum_{k=1}^n \lambda_k v_k$. It is a known fact from linear algebra. This and properties of linear systems imply that any solution $L(t)$ to a system of ODE $L' = A(t)L$ with initial data $L(0) = L_0$ can be represented as a linear combination $L(t) = \sum_{k=1}^n \lambda_k x_k(t)$ of solutions solutions $x_k(t)$ to the system of equations with initial data $x_k(0) = v_k$.

   Now to state that the space of solutions has dimension n we like to show that solutions $x_k(t)$ build a basis in the space of solutions. We have to show now that solutions $x_k(t)$ are linear independent. It means that according to the definition we must show that the relation $\sum_{k=1}^n \lambda_k x_k(t) = 0$ implies that all λ_k must be zero.

   Suppose that the linear combination $\sum_{k=1}^n \lambda_k x_k(t) = 0$ at some time point t. It means that the solution $L(t) = \sum_{k=1}^n \lambda_k x_k(t)$ coinsides with zero solution at this time point and because of the uniquness of solutions must be equal to zero at any time including $t = 0$. It means that $L(0) = L_0 = \sum_{k=1}^n \lambda_k v_k = 0$. But vectors v_k were chosen linearly independent. It implies that all λ_k must be zero. Therefore the dimension of the solutions space to $x' = A(t)x$ is equal to n.

2. Give definitions for a stable fixed point and an asymptotically stable fixed point to an autonomous system of ODE. Formulate and prove to the theorem about the asymptotic stability of a fixed point to an autonomous system of ODE using linearization. **(4p)**

   Answer. Check Theorem 4.2.1 on page 85and definitions 4.1.1 on page 81 in the book by Hsu.

3. Consider the following initial value problem: $y' = y^3 + t$; $y(1) = 1$.

   a) Reduce the problem to an integral equation. Calculate three first Picard approximations as in the proof to the Picard - Lindelöf theorem.

   b) Find some time interval where Picard approximations to this problem converge to the solution of the initial value problem. **(4p)**

   Solution.

   We introduce an integral equation equivalent to the ODE:

$$y(t) = y(1) + \int_1^t f(s, y(s))\mathrm{d}s$$

Taking $y_0(t) = y(1)$ we define Picard iterations by the recurrense relation

$$y_{n+1}(t) = y(1) + \int_1^t f(s, y_n(s))\mathrm{d}s$$

and compute them for the particular equation. For the particular equation it looks as

$$y_{n+1}(t) = y(1) + \int_1^t \left(y_n^3(s) + s\right) \mathrm{d}s$$

First three iterations are:

$$\begin{aligned}
y_0(t) &= 1 \\
y_1(t) &= 1 + \int_1^t (1+s)\,\mathrm{d}s = t + \frac{1}{2}t^2 - \frac{1}{2} \\
y_2(t) &= 1 + \int_1^t \left(\left(s + \frac{1}{2}s^2 - \frac{1}{2}\right)^3 + s\right)\mathrm{d}s = \frac{7}{8}t^2 - \frac{1}{8}t - \frac{3}{8}t^3 - \frac{1}{8}t^4 + \frac{9}{40}t^5 + \frac{1}{8}t^6 + \frac{1}{56}t^7 + \frac{107}{280}
\end{aligned}$$

They will converge on a time interval depending on the Lipschitz constant of the right hand side, essentially on the maximum of the $\left|\frac{\partial}{\partial y}(f(s,y))\right|$ on a properly chosen domain of y and t. We choose an interval for $y \in [0,2]$ and $t \in [1,2]$. We introduce notation $D = [1,2] \times [0,2]$.

$\frac{\partial}{\partial y}(f(s,y)) = 3y^2$. Convergence of the Picard iterations depends on the maximum of the absolute value of the right hand side of the equation and on its Lipschitz constant. The right hand side satisfy the estimate $M = \max_{(t,y)\in D}(f(t,y)) = 2^3 + 2 = 10$.

Therefore the solution satisfying initial conditions $y(1) = 1$ will not leave the domain $[0,2]$ around $y(0) = 1$ for $|t-1| < 1/M = 0.1$.

The Lipschitz constant L for $y \in [0,2]$ satisfies $L \le 12$ that is the maximum of $3y^2$ on the interval $y \in [0,2]$.

Picard iterations converge if conditions for the Banach contraction principle are satisfied, that for this problem requires conditions on the Lipschitz constant: $L\,|t-1| < 1$ or $|t-1| < 1/L = 1/12$. Comparing these two conditions for the time interval we choose the smaller interval $|t-1| < 1/12$.

4. Consider the following system of ODE:

$\frac{d\vec{r}(t)}{dt} = A\vec{r}(t)$, with a constant matrix $A = \begin{bmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$., characteristic polynomial: $X^3 + X^2 - X - 1 = (X-1)(X+1)^2$

Give a general real solution to the system. Find all those initial vectors $\vec{r_0} = \vec{r}(0)$ that give bounded solutions to the system. **(4p)**

Solution. $\begin{bmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$, eigenvectors: $v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \leftrightarrow \lambda_1 = -1$,is multiple eigenvalue $v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \leftrightarrow$ $\lambda_2 = 1$ - simple eigenvalue

Generalized eigenvector $v^{(1)}$ for $\lambda_1 = -1$ satisfies the equation $\begin{bmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

or $\begin{cases} 2x + y = -1 \\ y - 2z = 2 \end{cases}$ $z = 0,\ y = 2\,,\ x = -3/2,\ v^{(1)} = \begin{bmatrix} -3/2 \\ 2 \\ 0 \end{bmatrix}$.

General solution can be found in the general form:

$$x(t) = e^{At}x_0 = \sum_{j=1}^{s}\left(\left[\sum_{k=0}^{n_j-1}(A-\lambda_j I)^k\frac{t^k}{k!}\right]x^{0,j}e^{\lambda_j t}\right)$$

$x_0 = \sum_{j=1}^{s} x^{0,j}$ where $x^{0,j} \in M(\lambda_j, A)$ - are projections of x_0 to all generalized eigenspaces $M(\lambda_j, A)$ of the matrix A corresponding to each eigenvector.

We choose x_0 in the form $x_0 = C_1v_1 + C_2v_1^{(1)} + C_3v_2$ where $C_1v_1 + C_2v_1^{(1)} = x^{0,1} \in M(\lambda_1, A)$ and v_1 and $v_1^{(1)}$ constitute basis for $M(\lambda_1, A)$. $M(\lambda_2, A)$ has dimension one because the eigenvalue λ_2 is simple and $x^{0,2} = C_3v_2$ in the general formula above.

$x(t) = e^{At}x_0 = e^{\lambda_1 t}\left(C_1v_1 + C_2v_1^{(1)}\right) + te^{\lambda_1 t}(A-\lambda_j I)\left(C_1v_1 + C_2v_1^{(1)}\right) + C_3e^{\lambda_2 t}v_2 =$

$e^{\lambda_1 t}\left(C_1v_1 + C_2v_1^{(1)}\right) + te^{\lambda_1 t}(C_2v_1) + C_3e^{\lambda_2 t}v_2$. Substituting eigenvalues, eigenvectors and generalized eigenvectors for the particular problem we get the general solution:

$x(t) = C_1e^{-t}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + C_2e^{-t}\begin{bmatrix} -3/2 \\ 2 \\ 0 \end{bmatrix} + te^{-t}C_2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + C_3e^{t}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

Solutions with initial data orthogonal in the plane including vectors v_1 and $v_1^{(1)}$ be bounded and converge to the origin with $t \to +\infty$.

5. Consider the following linear system of ODE with periodic coefficients:

$\dfrac{d\overrightarrow{r}(t)}{dt} = A(t)\overrightarrow{r}(t)$, with matrix $A(t) = (a + \sin^2(t))\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Use Floquet theory to find for which real constants a its solutions are bounded.

Hint: make a change of the time variable to find a monodromy matrix. **(4p)**

Consider the equation in the form

$$\frac{1}{(a+\sin^2(t))}\frac{d\overrightarrow{r}(t)}{dt} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\overrightarrow{r}$$

Introduce anew time variable $\tau(t) = \int_0^t (a+\sin^2(s))ds$. The change of variables in time differentiation will be

$$\frac{d}{dt} = \frac{d\tau(t)}{dt}\frac{d}{d\tau} = \frac{d\left(\int_0^t (a+\sin^2(s))ds\right)}{dt}\frac{d}{d\tau} = (a+\sin^2(t))\frac{d}{d\tau}$$

$$\int_0^t \sin^2(s)ds$$

$\frac{1}{2}t + at - \frac{1}{4}\sin 2t$

Therefore

$$\frac{1}{(a+\sin^2(t))}\frac{d\overrightarrow{r}(t)}{dt} = \frac{d\overrightarrow{r}(\tau)}{d\tau} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\overrightarrow{r} = B\overrightarrow{r}$$

and we have got a linear system of ODEs with constant coefficients in terms of the τ variable and can solve it exactly. Its principal matrix is $\exp(\tau B)$ and

$$r(\tau) = \exp(\tau B)r(0)$$

with $\exp(0B) = I$.

Principal matrix for the original system is $\exp(\tau(t)B)$ with $\tau(t) = \int_0^t (a + \sin^2(s))ds$ and we observe that $\exp(B\tau(0)) = I$. The monodromy matrix of the original system will be

$$M = \exp\left(\tau(2\pi)B\right)$$

Eigenvalues of the matrix B are $\lambda_1 = \frac{3}{2} - \frac{1}{2}\sqrt{5}$ and $\lambda_2 = \frac{1}{2}\sqrt{5} + \frac{3}{2}$ - both positive. Floquet multypliers are $\exp(\lambda_1\tau(2\pi))$ and $\exp(\lambda_2\tau(2\pi))$. Floquet exponents are evidently $\frac{1}{2\pi}\left(\lambda_1\tau(2\pi)\right)$ and $\frac{1}{2\pi}\left(\lambda_2\tau(2\pi)\right)$.

We must have $\tau(2\pi) \le 0$ to have the both Floquet exponents non-positive and correspondingly to have Floquet multypliers not larger than 1.

It will imply by the Floque theorem that solutions to the given system of ODE will be bounded because $\lambda_1\tau(2\pi)$ and $\lambda_2\tau(2\pi)$ are different (not multiple). Checking the values of the integral $\tau(2\pi) = \int_0^{2\pi}(a + \sin^2(s))ds = \pi + 2\pi a$ we observe that to have $\tau(2\pi) \le 0$, a must satisfy the inequality $a \le -1/2$. The same idea would in fact work for any function instead of $(a + \sin^2(s))$ in the definition of $A(t)$.

6. Show that the following system of ODE has a periodic orbit.

$$\begin{cases} x' = x - y - x^3 \\ y' = x + y - y^3 \end{cases} \qquad \textbf{(4p)}$$

Choose a simple positive definite test function $V(x,y) = \frac{x^2+y^2}{2}$

$V'(x,y) = x^2 + y^2 - x^4 - y^4 = x^2(1-x^2) + y^2\left(1-y^2\right)$

We observe easily that $V'(x,y) > 0$ for $|x| < 1$ and $|y| < 1$. It means that the trajectory starting outside the circle $x^2 + y^2 = 1$ will never enter it.

On the other hand $V'(x,y) < 0$ for $|x| > 2$ and $|y| > 2$. It means that the trajectory starting inside the circle $x^2 + y^2 = 8$ will never leave it.

Therefore the annulus $1 \le x^2 + y^2 \le 4$ is a positively invariant set of the system (solutions will be eternal because the right hand side is a C^1 function)

The system has the only stationary point in the origin outside this annulus. We observe it by checking together zeros of $V'(x,y)$ and zeros of the right hand sides in the equations.

Therefore by the Bendixson theorem there must be at least one periodic orbit there.

Max. 24 points;

Thresholding for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course. Total points for the course are calculated as $Total = 0.3Assignments + 0.7Exam$ - the average of the points for the home assignments (30%) and for this exam (70%). The same thresholding is valid for the exam, for the home assignments, and for the total points for the course.