MATEMATIK	Datum: 2018-05-28	Tid: 08-30 - 12-30
GU, Chalmers	Hjälpmedel: - Inga	
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Lösningsförslag till tenta i ODE och matematisk modellering, MMG511, MVE162 (MVE161)

1. Formulate and prove the theorem about properties of solutions to linear systems of ODEs with periodic coefficients: $x' = A(t)x$; $A(t+p) = A(t)$, when $t \to \infty$. **(4p)**

Theorem on boundedness and zero limits of solutions to periodic linear systems.

1) Every solution to a periodic linear system is bounded on $\mathbb{R}_+$ if and only if the absolute value of each Floquet multiplier is not greater than 1 and any Floquet multiplier with absolute value 1 is semisimple.

2) Every solution to a periodic linear system tends to zero at $t \to \infty$ if and only if the absolute value of each Floquet multiplier is strictly less than 1.

Floquet multipliers are eigenvalues to the monodromy matrix $\Phi(p,0)$ of the system.

By Floquet theorem any solution $x(t)$ to system

$$x'(t) = A(t)x(t), \quad A(t+p) = A(t), \forall t \in \mathbb{R} \tag{1}$$

satisfying initial conditions $x(\tau) = \xi$, is represented as

$$x(t) = \Phi(t,\tau)\xi = \Theta(t)\exp(tF)\Phi(0,\tau)\xi = \Theta(t)\exp(tF)\zeta$$

where $F = \frac{1}{p}\mathrm{Log}(\Phi(p,0)), \Phi(0,\tau)\xi = \zeta$. $\Theta(t)$ is a p - periodic continuous or piecewise continuous matrix valued function. $\Theta(t)$ is invertible for all t as a product of invertible matrices.

We define $y(t) = \exp(tF)\zeta$ as a solution to the equation

$$y'(t) = F\,y, \quad y(0) = \zeta \tag{2}$$

$y(t) = \Theta^{-1}(t)x(t)$, and $x(t) = \Theta(t)y(t)$. The mapping $\Theta(t)$ determines a one to one correspondence between solutions to the periodic system (1) and the autonomous system (2).The periodicity and continuity properties of $\Theta(t)$ and $\Theta^{-1}(t)$ imply that there is a constant $M > 0$ such that $\|\Theta(t)\| \leq M$ and $\left\|\Theta^{-1}(t)\right\| \leq M$ for all $t \in \mathbb{R}$. It implies that $\|x(t)\| \leq M\,\|y(t)\|$ and $\|y(t)\| \leq M\,\|x(t)\|$.

Therefore

1)$\|x(t)\|$ is bounded on $\mathbb{R}_+$ if and only if corresponding $\|y(t)\| = \|\exp(tF)\zeta\|$ is bounded on $\mathbb{R}_+$.

2) $\|x(t)\| \to 0$ when $t \to \infty$ if and only if corresponding $\|y(t)\| \to 0$ when $t \to \infty$.

Since $\mathrm{Log}\,(\Phi(p,0)) = G = pF$, it follows that

$$\begin{aligned} \sigma\,(\Phi(p,0)) &= \{\exp(\lambda p) : \quad \lambda \in \sigma(F)\} \\ \sigma\,(F) &= \left\{\frac{1}{p}\mathrm{Log}(\mu) : \mu \in \sigma(\Phi(p,0))\right\} \end{aligned}$$

and that algebraic and geometric multiplicities of each $\lambda \in \sigma(F)$ coincide with those of $\exp(p\lambda) \in \sigma\,(\Phi(p,0))$ .We use now that

$$\begin{aligned} \mathrm{Log}(z) &= \ln(|z|) + i\arg(z) \\ \exp(z) &= \exp(\mathrm{Re}\,z)(\cos(\mathrm{Im}\,z) + i\sin(\mathrm{Im}\,z) \end{aligned}$$

Here we choose the argument $\arg(z) \in [0.2\pi)$ to make Log an invertible function that is called the principle logarithm.

The following connections between properties of Floquet multipliers and properties of corresponding eigenvalues to the matrix $F = \frac{1}{p}\mathrm{Log}(\Phi(p,0))$ are a direct consequence:

a) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$,has$|\mu| < 1$ if and only if $\mathrm{Re}\,\mathrm{Log}(\mu) < 0$ that is if the corresponding eigenvalue $\lambda = \frac{1}{p}\mathrm{Log}(\mu)$ to F has $\mathrm{Re}\,\mathrm{Log}(\mu) < 0$.

b) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$,has $|\mu| \leq 1$ if and only if $\mathrm{Re}\,\mathrm{Log}(\mu) \leq 0$ that is if the corresponding eigenvalue $\lambda = \frac{1}{p}\mathrm{Log}(\mu)$ to F has $\mathrm{Re}\,\mathrm{Log}(\mu) \leq 0$.

c) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$, with $|\mu| = 1$ is semisimple if and only if the corresponding eigenvalue $\lambda = \frac{1}{p}\mathrm{Log}(\mu)$ to F has $\mathrm{Re}\,\mathrm{Log}(\mu) = 0$ is semisimple.

Known relations between properties of solutions to an autonomous system and the spectrum of corresponding matrix applied to the system $y'(t) = F\,y$ and to the spectrum $\sigma(F)$ of the matrix F together with statements 1),2), a),b),c) in the present proof imply the statement of the theorem.■

2. Formulate and give a proof to Lyapunov's theorem on stability of equilibrium points. **(4p)**

Lyapunov's theorem on stability

Let 0 be an equilibrium point for the system $x' = f(x)$ with $f : G \to \mathbb{R}^n$, Lipschitz, $f(0) = 0$. Suppose there is a positive definite continuously differentiable, $C^1(U)$ function $V : U \to \mathbb{R}$, such that $U \subset G$, open, $0 \in U$ and $V_f(z) = \nabla V \cdot f(z) \leq 0 \; \forall z \in U$. Then 0 is a stable equilibrium point.

Remark.

A function V with these properties is usually called the Lyapunov function of the system.

Proof.

Take an arbitrary $\varepsilon > 0$ such that $B(\varepsilon, 0) \subset U$. Let $\alpha = \min_{z \in S(\varepsilon,0)} V(z)$ be a minimum of the continuous function V on the boundary of $B(\varepsilon, 0)$, that is the sphere $S(\varepsilon, 0) = \{z : \|z\| = \varepsilon\}$ and is a compact set (closed and bounded). Then $\alpha > 0$ because $V(z) > 0$ outside the equilibrium point 0.

By continuity of the function V and the fact that $V(0) = 0$ one can find a $0 < \delta < \varepsilon$ such that $\forall z \in B(\delta, 0)$ we have $V(z) < \alpha/2$.

On the other hand for any part of the trajectory $x(t) = \varphi(t, \xi)$, inside U the function $V(\varphi(t, \xi))$ is *non-increasing* because $\frac{d}{dt} V(\varphi(t, \xi)) = (\nabla V \cdot f)(x(t)) \leq 0$. Therefore all trajectories $\varphi(t, \xi)$ with initial points $\xi \in B(\delta, 0)$ satisfy $V(\xi) < \alpha/2$. Therefore $V(\varphi(t, \xi)) < \alpha/2$ and $\varphi(t, \xi)$ cannot reach the sphere $S(\varepsilon, 0)$ where $V(z) \geq \alpha = \min_{z \in S(\varepsilon,0)} V(z)$. Therefore any such trajectory stays within the ball $B(\varepsilon, 0)$ and by the definition, the origin 0 is stable. It implies also that $\mathbb{R}^+ \subset I_\xi$, where I_ξ is the maximal interval for initial point ξ, because the trajectory stays inside a compact set. ■

3. Consider the following system of ODE: $\dfrac{d\overrightarrow{r}(t)}{dt} = A\overrightarrow{r}(t)$, with a constant matrix A having eigenvalues 3 and 0:

$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix}$. Find a general solution to the system and a canonical Jordan form of the matrix A. Find all initial conditions such that solutions to I.V.P. will be bounded. **(4p)**

Solution.

We find eigenvectors corresponding eigenvalues.

$\lambda = 0$.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 2 & 1 & 0 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

an eigenvector can be taken as $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.There are no other linearly independent eigenvectors corresponding to the eigenvalue $\lambda = 0$.

$\lambda = 3$

$$A - 3I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 4 \\ 0 & 1 & -4 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

an eigenvector can be taken as $v_2 = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$

We check if $\lambda = 0$, has an generalized eigenvector, namely if $Av_1^{(1)} = v_1$ has a solution.

$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 4 & -2 \\ 1 & 0 & -1 & 1 \end{bmatrix} \overset{Gauss}{\Longrightarrow} \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, reduced row echelon form is: $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.Generalized eigenvector can be chosen as $v_1^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Vectors v_1, $v_1^{(1)}$, and v_2 build a basis in $\mathbb{R}^3$.

The canonical Jordan form for the matrix A is $J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

General solution to the system with initial data $\xi = C_1 v_1 + C_2 v_1^{(1)} + C_3 v_2$is

$$x(t) = C_1 v_1 + tC_2 v_1 + C_2 v_1^{(1)} + e^{3t} C_3 v_2$$

Solutions are bounded if and only if $\xi = C_1 v_1$

4. Find maximal existence time interval for solution of the I.V.P. $x' = 2t\,x^2$ with arbitrary initial data $x(0) = \xi$. **(4p)**

$\frac{dx}{dt} = 2t\,x^2$, is the equation with separable variables.

$\int \frac{dx}{x^2} = \int 2t dt\,;\ \frac{-1}{x} = t^2 + C_1$; general solution is: $x(t) = \frac{1}{-C_1 - t^2}$.

To satisfy the initial condition we put $t = 0$ and find $x(0) = \xi = \frac{1}{-C_1}$, therefore $C_1 = -\frac{1}{\xi}$ and $x(t) = \frac{1}{\frac{1}{\xi} - t^2}$.

$\xi = 0$ gives a constant solution $x(t) = 0$ for all $t \in \mathbb{R}$. $I_{\max}(\xi) = \mathbb{R}$

For $\xi < 0$ we observe that the solution $x(t) = \frac{1}{\frac{1}{\xi} - t^2}$ is defined for all $t \in \mathbb{R}$. Therefore $I_{\max}(\xi) = \mathbb{R}$.

For $\xi > 0$ we observe that the solution $x(t) = \frac{1}{\frac{1}{\xi} - t^2}$ is defined only for $t \in (\frac{-1}{\sqrt{\xi}}, \frac{1}{\sqrt{\xi}})$. The solution blows up in the points $\pm\frac{1}{\sqrt{\xi}}$. Therefore in this case $I_{\max}(\xi) = (\frac{-1}{\sqrt{\xi}}, \frac{1}{\sqrt{\xi}})$.

5. Consider the following system of ODEs representing a one-dimensional mechanical system with velocity y and coordinate x. Find all equilibrium points, investigate their stability properties, and find their possible domains of attraction.

$$\begin{cases} x' = & y \\ y' = & -y - x - x^2 \end{cases} \qquad \textbf{(4p)}$$

Equilibrium points are $(0, 0)$ and $(-1, 0)$.

Jacobi matrix is $A(x, y) = \begin{bmatrix} 0 & 1 \\ -1 - 2x & -1 \end{bmatrix}$

Jacobi matrix at the origin is $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$,

characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda + 1$, eigenvalues are $-\frac{1}{2}i\sqrt{3} - \frac{1}{2}, \frac{1}{2}i\sqrt{3} - \frac{1}{2}$. Real parts of eigenvalues are negative and therefore the origin is stable focus, asymptotically stable equilibrium.

Jacobi matrix at the point $(-1, 0)$ is $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$,

characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda - 1$,eigenvalues are $-\frac{1}{2}\sqrt{5} - \frac{1}{2}, \frac{1}{2}\sqrt{5} - \frac{1}{2}$. One is negative, another is positive, the equilibrium point is a saddle point and is unstable.

We try to find the domain of attraction for the asymptotically stable point in the origin.

The system represents the one dimensional Newton equation with potential force with potential $G(x) = \int_0^x (z + z^2)\, dz = \frac{1}{2}x^2 + \frac{1}{3}x^3$. A natural Lyapunov function to such a system is the sum of kinetic and potential energy:

$$V(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3$$

Calculating derivative of V along trajectories we arrive to

$$\begin{aligned} V_f(x, y) &= \nabla V \cdot f = \left(x + x^2\right)(y) + y\left(-y - x - x^2\right) \\ &= -y^2 \leq 0 \end{aligned}$$

We can apply LaSalle's invariance principle to find domain of attraction of the equilibrium in the origin. The set $V_f^{-1}(0)$ denotes the set where $V_f(x, y) = 0$. If a trajectory $\varphi(t, \xi)$ has the closure of the positive semi orbit $O^+(\xi)$ compact and contained in the open set U where $V_f(x, y) \leq 0$, then $\varphi(t, \xi)$ approaches the union of invariant sets (orbits) in $V_f^{-1}(0)$ as $t \to \infty$. If this union of invariant sets consists of one point, then the trajectory $\varphi(t, \xi)$ tends to this point as $t \to \infty$.

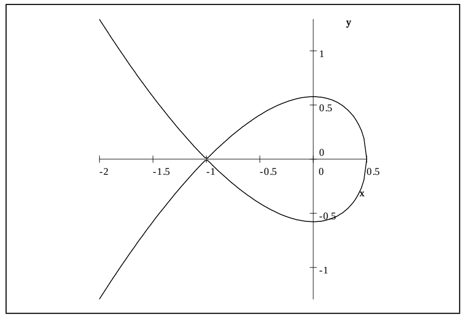
This condition on the trajectory $\varphi(t, \xi)$ is easy to check by finding a compact positively invariant set K for the system where the condition $V_f(x, y) \leq 0$ is satisfied.Then all trajectories starting in K will have the closure of the positive semi- orbit $O^+(\xi)$ compact and contained in K. If this positively invariant set includes only one equilibrium as an invariant set in $V_f^{-1}(0)$, then this equilibrium is an attractor for all trajectories starting in K that is it's domain of attraction.

Positively invariant sets of our system can be found as sets bounded by level sets of the Lyapunov function $V(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3$ (with not very large values to have them as closed curves). We need to find one such positively invariant set that does not contain the equilibrium point $(-1, 0)$. The largest such set would be one with boundary going exactly through the point $(-1, 0)$.

Taking the level set of V going through this point we get corresponding value for V: $V(-1, 0) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

The level set $V(x, y) = \frac{1}{6}$ can be expressed by the implicit equation $\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 = \frac{1}{6}$ and is evidently symmetric with respect to x axis. We need to investigate it's shape.

Considering the derivative $\frac{d}{dx}\left(-\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{6}\right) = -x - x^2$ of the function $g(x) = -\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{6}$ we observe that $g(x)$ is rising between -1 and 0, $g(-1) = 0$, $g(0) = \frac{1}{6}$, and $g(x)$ is decreasing for $0 < x$. In the point $x = 0.5$ it is equal to zero. Therefore for $x \in [-1, 0.5]$ the function $g(x)$ is non-negative. It implies that for $x \in [-1, 0.5]$ the level set $V(x, y) = \frac{1}{6}$ consists of two curves symmetric with respect to x axis: $y = \pm\sqrt{-x^2 - \frac{2}{3}x^3 + \frac{1}{3}}$. It gives the boundary of the bounded domain U symmetric with respect to the $x-$axis. The part of the level set $V(x, y) = \frac{1}{6}$ in the half plane $x < -1$ does not bound a compact set and is of no interest for consideration of the equilibrium in the origin.



This domain is positively invariant because $V_f(x,y) \le 0$ for all (x,y). The same is valid also for any smaller set bounded by level sets $V(x,y) = C < 1/6$.

Points $(x,y) \in U$ where $V_f(x,y) = 0-$ are on the x - axis. Except the origin, they all have velocities $f(x,y)$ pointed across the x- axis and cannot belong to an invariant set in $V_f^{-1}(0)$.The second equilibrium point $(-1,0)$ of the system was excluded before by the choice of the level set of $V(x,y) = 1/6$. Therefore the origin is the only invariant set in $V_f^{-1}(0) \cap U$. It implies that all trajectories starting in U tend to the origin as $t \to \infty$, and therefore U is the domain of attraction for the asymptotically equilibrium point in the origin.

6. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = y \\ y' = -x - y\ln\left(x^2 + 4y^2\right) \end{cases} \qquad \textbf{(4p)}$$

We will consider a simple test function $V(x,y) = \frac{1}{2}(x^2 + y^2)$.

$V_f(x,y) = xy - xy - y^2 \ln\left(x^2 + 4y^2\right) = -y^2 \ln\left(x^2 + 4y^2\right)$.

The ellipse $x^2 + 4y^2 = 1$ (red in the picture) is the boundary between the domains where $V_f(x,y)$ has negative and positive values.

For $x^2 + 4y^2 \le 1$ we get $V_f(x,y) \ge 0$

For $x^2 + 4y^2 > 1$ we get $V_f(x,y) < 0$

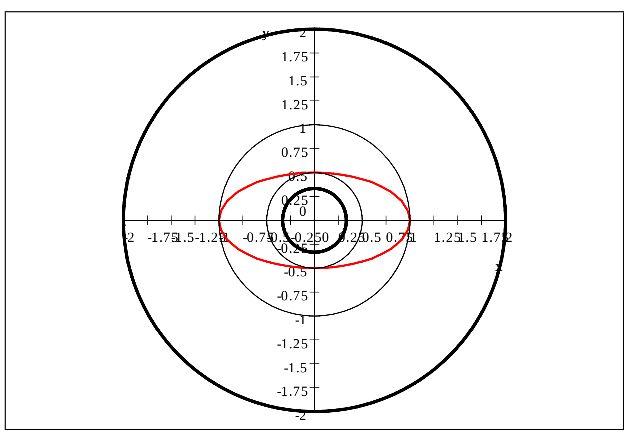
Therefore a circle with radius r so small that $x^2 + 4y^2 \le 1$ for $x^2 + y^2 < r^2$ will have the property that no one trajectory enters it.

Therefore a circle with radius R so large that $x^2 + 4y^2 > 1$ for $x^2 + y^2 \ge R^2$ will have the property that no one trajectory escapes it.

The largest circle inside the ellipse $x^2 + 4y^2 = 1$ is $4x^2 + 4y^2 = 1$, or $x^2 + y^2 = \left(\frac{1}{2}\right)^2$

The smallest circle outside the ellipse $x^2 + 4y^2 = 1$ is $x^2 + y^2 = 1$.

$x^2 + 4y^2 - 1 = 0$



Therefore choosing $r = \frac{1}{3}$ and $R = 2$ we observe that these two circles bound an annulus shaped set $K = \left\{(x,y) : \quad r^2 \le x^2 + y^2 \le R^2\right\}$ that is positively invariant and compact. The system has no equilibrium in this set. Therefore according to the Poincare-Bendixson theorem K must contain at least one periodic orbit.

Max. 24 points;

Threshold for marks: for GU: **VG:** 19 points; **G:** 12 points. For Chalmers: **5:** 21 points; **4:** 17 points; **3**: 12 points;

One must pass both the home assignments and the exam to pass the course.

Total points for the course are calculated as:

$Total = 0.16\, Assignment1 + 0.16\, Assignment2 + 0.68\, Exam$ - that is the average of the points for the home assignments (32%) and for this exam (68%). The same threshold is valid for the exam, for the home assignments, and for the total points for the course.

Points that you have got for the assignments and for the exam are valid and are kept up to the moment when you will collect all necessary points.