6. Formulate the Poincare-Bendixson theorem and use it to show that the following system of ODE has a periodic solution $(x(t), y(t)) \neq (0,0)$.

$$\begin{cases} x' = & y \\ y' = & -f(x,y)\,y - x \end{cases}$$

where f, f'_x , f'_y are continuous, $f(0,0) < 0$ and $f(x,y) > 0$ for $x^2 + y^2 > b^2$.

(4p)

One of formulations of the Poincare Bendixson theorem is: let $\varphi(t)$ be a solution to an autonomous equation $x' = f(x)$ in plane, bounded for all $t > 0$. We suppose that f is Lipschitz to guarantee uniquness of solutions. Then the limit set $\omega(\varphi)$ of $\varphi(t)$ has the following property: it either

i) contains an equilibrium

or

ii) $\varphi(t)$ is periodic itself

or $\omega(\varphi)$ is a periodic orbit.

The theorem cas an important corollary, that is also often called Poincare Bendixson theorem:

If the equation $x' = f(x)$ in plane has a compact positively invariant set (a set that no trajectories can leave) that does not include any equilibrium points, then this set must include at least one periodic orbit.

We apply the corollary to the example above and try to find a set in plane satisfying requirements i the corollary.

We observe first that solutions exist for initial points everywhere in the plane because of the smoothness of f.

Consider a test function $V(x,y) = \left(x^2 + y^2\right)/2$ and its evolution along solutions to the given system.

$V' = xy - y^2 f(x,y) - xy = -y^2 f(x,y)$. $f(x,y) > 0$ for $x^2 + y^2 > b^2$. It implies that $V' \leq 0$ for $x^2 + y^2 = b^2$ and that trajectories of the system cannot leave the disc $x^2 + y^2 \leq b^2$.

$f(0,0) < 0$ and is continuous. It implies that there is a small circle around the origin $x^2 + y^2 \leq \delta^2$

such that $f(x,y) < 0$ and correspondingly $V' > 0$ for $x^2 + y^2 = \delta^2$. It implies that the system cannot leave the ring $\delta^2 \leq x^2 + y^2 \leq b^2$. It is easy to observe that the only equilibrium point of the system is the origin: $y = 0$ for the first equation and therefore $x = 0$ from the second equation.

Alltogether implies that the ring $\delta^2 \leq x^2 + y^2 \leq b^2$ is a positively invariant set without equilibrium points and must include at least one periodic orbit.

iii) Formulate the Poincare- Bendixon theorem and use it to show that the following system of ODE has an ω - limit set that is a periodic orbit.

$$\begin{cases} x' = & -x\left(x^2+y^2-3x-1\right)+y \\ y' = & -y\left(x^2+y^2-3x-1\right)-x \end{cases} \qquad \textbf{(4p)}$$

Solution.

i) $\omega(p)$ is an ω -limit set of a point p for a continuous dynamical system $\pi(t,x)$ if for any point $z \in \omega(p)$ there is a sequence of times $\{t_n\}_{n=1}^{\infty}$ depending on z, such that $t_n \underset{n\to\infty}{\longrightarrow} \infty$ and $\pi(t_n, p) \underset{n\to\infty}{\longrightarrow} z$.

ii) If the trajectory $\pi([0,\infty], x)$ of the dynamical system is bounded, then the ω -limit set $\omega(p)$ is not empty, closed, connected and positively (or ω) - invariant set.

iii) Poincare Bendixson theorem states that if an ODE in plane has a positively invariant set U without equilibrium points then it must include a periodic orbit that is an ω - limit set for all points in thes set U.

We try to use a simple test function $V(x,y) = \frac{1}{2}\left(x^2+y^2\right)$ to localize a positively invariant set without equilibrium points.

$V'(x(t),y(t)) = -x^2\left(x^2+y^2-3x-1\right)+xy+-y^2\left(x^2+y^2-3x-1\right)-xy = -\left(x^2+y^2\right)\left(x^2+y^2-3x-1\right)$

It implies that for large enough x^2+y^2 we have $V'(x(t),y(t)) \leq 0$ and for small enough $x^2+y^2 \neq 0$ we have $V'(x(t),y(t)) > 0$.

Therefore there are δ and R, $0<\delta<R$ such that the ring $\delta \leq \sqrt{x^2+y^2} \leq R$ is a positively invariant set and therefore must include at least one periodic orbit that is an ω - limit set for all points in this ring.

6. Formulate Poincare-Bendixson theorem. Find a positively invariant set for the following system of ODE. Show that the system has at least one periodic solution.

$$\begin{cases} x' = & -y/3+x(1-3x^2-y^2) \\ y' = & x+y(1-3x^2-y^2) \end{cases} \qquad \textbf{(4p)}$$

Poincare-Bendixson theorem. Consider a system $r' = f(r)$ in the plane R^2. If a limit set $\omega_\sigma(r)$ of a point r is not empty, compact and contains no fixed points, it is a regular periodic orbit.□

A corollary of the theorem is that if C is a compact positively invariant set to a system of ODE in the plane and C does not contain any fixed points, it must contain at least one regular periodic orbit.□

Multiply the first equation by $3x$ and the second equation by y and add:

$\frac{1}{2}\left(3x^2+y^2\right)' = \left(3x^2+y^2\right)\left(1-\left(3x^2+y^2\right)\right)$

The function $V(x,y) = 3x^2+y^2$ satisfies the equation: $V'(t) = 2V(1-V)$.

We observe that $V(t)$ increases along trajectories of the system for $V<1$ and V decreases for $V>1$. It implies that the set $G = \left\{(x,y) : 0.5 \leq 3x^2+y^2 \leq 2\right\}$ (an elliptic ring round the origin) is a positively invariant set.

The same calculation shows that the origin is the only stationary point, because stationary point

must satisfy the equation $V(1-V)=0$. By inserting $1-3x^2-y^2=0$ into the equations one can see that points on the ellips $3x^2-y^2=1$ are not stationary, because thay must at the same time be in the origin. It leaves the only fixed point in the origin.

The corollary to Poincare Bendixson theorem states that in a compact positively invariant set without fixed points there must be at least one periodic solution.

4. **Periodic solutions to ODE.**

Show that the following system of ODE has a periodic solution.

$$\begin{cases} x' = y \\ y' = -x + y(1 - 3x^2 - 2y^2) \end{cases}$$

Hint: transform the system to polar coordinates and consider the equation for polar

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Expressing the system in polar coordinates r, θ we get:

$r' = r\sin^2(\theta)(1 - 3r^2\cos^2(\theta) - 2r\sin^2(\theta))$

We observe that for small enough $r \quad r' \geq 0$,

for example for $r = 0.5$: $r' = 0.25\sin^2(\theta)(1 - 0.5\cos^2(\theta)) \geq 0$

One observes also from the equation for r' that

$r' \leq r\sin^2(\theta)(1 - 2r^2)$ that makes $r' \leq 0$ for $r \leq 1/\sqrt{2}$. Equality is attained only for $\theta = 0, \theta = \pi$.

It makes the ring $0.5 < r < 1/\sqrt{2}$ a positively invariant set for the system.

The only fixpoint of the sistem is the origin, therefore by the Poincare-Bendixson theorem it must have a periodic solution in this ring.

Nonexistence of periodic solutions

6. Show that the following system of ODE has no periodic solutions.

$$\begin{cases} x' = \frac{1}{7} + x^2 - yx + y^2 \\ y' = -\frac{1}{5} - y^2 \end{cases} \qquad \textbf{(4p)}$$

Solution

y' is always strictly negative. It implies that $y(t)$ must be monotone function of time. It contradicts to possibility of having periodic solutions that are always bounded.

6. Show that the following system of ODE has no periodic solutions.

$$\begin{cases} x' = x^3 - y^2x + x \\ y' = -0.5y + y^3 + x^4y \end{cases} \qquad \textbf{(4p)}$$

We consider divergence of the right hand side of the system.

$div(f) = 3x^2 - y^2 + 1 - 0.5 + 3y^2 + x^4 = x^4 + 3x^2 + 2y^2 + 0.5 > 0$

Therefore divergence of the right hand side of the equation is positive everywhere in the plane that is a simply connected set (does not have holes). According to Bendixson's criterion the system cannot have periodic solutions anywhere in the plane.