1 Stability of equilibrium points by linearization.

1.1 Definition of stable equilibrium points.

We consider in this chapter of the course properties of solutions I.V.P to nonlinear autonomous systems of ODE

$$x' = f(x), \quad x(0) = \xi \tag{1}$$

where $f : G \to \mathbb{R}^N$ is locally Lipschitz with respect to x. J is and interval and $G \subset \mathbb{R}^N$ is a non-empty open set.

Definition. (p. 115, L.R.) A function f is called locally Lipschitz in G if for any point $y \in G$ there is a neighbourhood $V(y)$ and a number $L > 0$ (depending on $V(y)$) such that for any $v, w \in V(y)$

$$\|f(v) - f(w)\| \leq L \|v - w\|$$

Example. Functions having continuous partial derivatives are locally Lipschitz function. (Exercise)

Definition. A solution $x(t) : I \to \mathbb{R}^N$ is called maximal solution if it cannot be extended to a larger time interval.

Claim (important!)

We will formulate later a theorem by Picard and Lindelöf, that implies that under these conditions the I.V.P. above has a unique solution for any $\xi \in G$ on some, might be small time interval $(-\delta, \delta)$. (Theorems 4.17, p. 118; Theorem 4.22, p.122.

Definition. A point $x_* \in G$ is called an equilibrium point to the equation (1) if $f(x_*) = 0$.

The corresponding solution $x(t) \equiv x_*$ is called an equilibrium solution.

Definition. (5.1, p. 169, L.R.)

The equilibrium point x_* is said to be stable if, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for any maximal solution $x : I \to G$ to (1) such that $0 \in I$ and $\|x(0) - x_*\| \leq \delta$ we have $\|x(t) - x_*\| \leq \varepsilon$ for any $t \in I \cap \mathbb{R}_+$. Below a picture is given in the case $x_* = 0$.

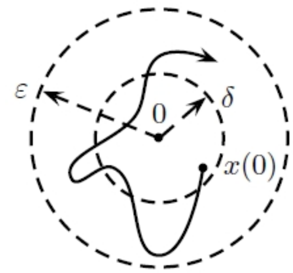


Figure 5.1 Stable equilibrium

Definition. (5.14, p. 182, L.R.)

The equilibrium point x_* of (1) is said to be *attractive* if there is $\delta > 0$ such that for every $\xi \in G$ with $\|\xi - x_*\| \leq \delta$ the following properties hold: the solution $x(t) = \varphi(t, \xi)$ to I.V.P. with $x(0) = \xi$ exists on $\mathbb{R}_+$and $\varphi(t, \xi) \to x_*$ as $t \to \infty$.

Definition. We say that the equilibrium x_* is asymptotically stable if it is both stable and attractive.

In the analysis of stability we will always choose a system of coordinates so that the origin coinsides with the equilibrium point. In the course book this agreement is applied even in the definition of stability.

Definition. The equilibrium point x_* is said to be *unstable* if it is not stable. It means that there is a $\varepsilon_0 > 0$, such that for any $\delta > 0$ there is point $x(0) : \|x(0) - x_*\| \leq \delta$ such that for some $t_0 \in I$ we have $\|x(t_0) - x_*\| > \varepsilon_0$.(a formal negation to the definition of stability)

1.2 Stability and unstability of the equilibrium point in the origin for autonomous linear systems.

Origin is an equilibrium point for all linear systems of ODE. If the matrix A is degenerate, there can be even lines or hyperplanes of equilibrium points except the origin, corresponding to the non-trivial kernel of the matrix A.

Example. Consider the system $x'(t) = Ax(t)$ with $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. Eigenvalues of the matrix A are $\lambda = \pm 2i$ are purely imaginary (and non-zero). Therefore there are no other equilibrium points except the origin. The $\exp(At) = \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix}$. The solution to the initial value problem with initial data $[\xi_1, \xi_2]^T$ is

$$\begin{aligned} x(t) &= \begin{bmatrix} \xi_1 \cos(2t) - \xi_2 \sin(2t) \\ \xi_1 \sin(2t) + \xi_2 \cos(2t) \end{bmatrix} = |\xi| \begin{bmatrix} \frac{\xi_1}{|\xi|} \cos(2t) - \frac{\xi_2}{|\xi|} \sin(2t) \\ \frac{\xi_1}{|\xi|} \sin(2t) + \frac{\xi_2}{|\xi|} \cos(2t) \end{bmatrix} = \\ &= |\xi| \begin{bmatrix} \cos(\theta)\cos(2t) - \sin(\theta)\sin(2t) \\ \cos(\theta)\sin(2t) + \sin(\theta)\cos(2t) \end{bmatrix} = |\xi| \begin{bmatrix} \cos(\theta + 2t) \\ \sin(\theta + 2t) \end{bmatrix} \end{aligned}$$

with $\cos(\theta) = \frac{\xi_1}{|\xi|}$. Therefore orbits of solutions are circles around the origin with the radius equal to $|\xi|$. It implies that the equlibrium point in the origin is stable. ε in the definition of stability can be chosen equal to δ.

Example.

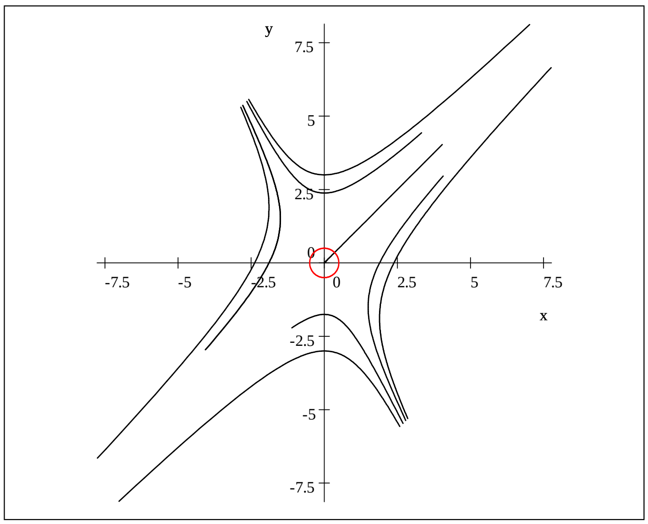
An example on instability: saddle point. There are trajectories (not all) that leave a neighbourhood $\|x\| < d$ of the origin for initial conditions ξ arbitrary close to the origin: for any $\varepsilon > 0$ and $0 < \|\xi\| \leq \varepsilon$ after some time T_ε.

$r' = Ar$ with $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - \lambda - 2 = 0$;

eigenvectors: $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \leftrightarrow -1, \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 2$

$$r = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

choosing a ball $\|x\| \leq 1$, and for arbitrary $\varepsilon > 0$, $\xi = \varepsilon \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\|\xi\|$ we see that the corresponding solution $x(t) = e^t \varepsilon \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ will leave this ball after time $T_\varepsilon = -\ln \varepsilon$.



A classification of phase portraits for non-degenerate linear autonomous systems in plane in terms of the determinant and the trace of the matrix A.

Stable (unstable) nodes when eigenvalues λ_1, λ_2 are real, different, negative (positive). $\det(A) < \frac{1}{4}(tr(A))^2$; $\det(A) > 0$; $tr(A) < 0$, ($tr(A) > 0$).

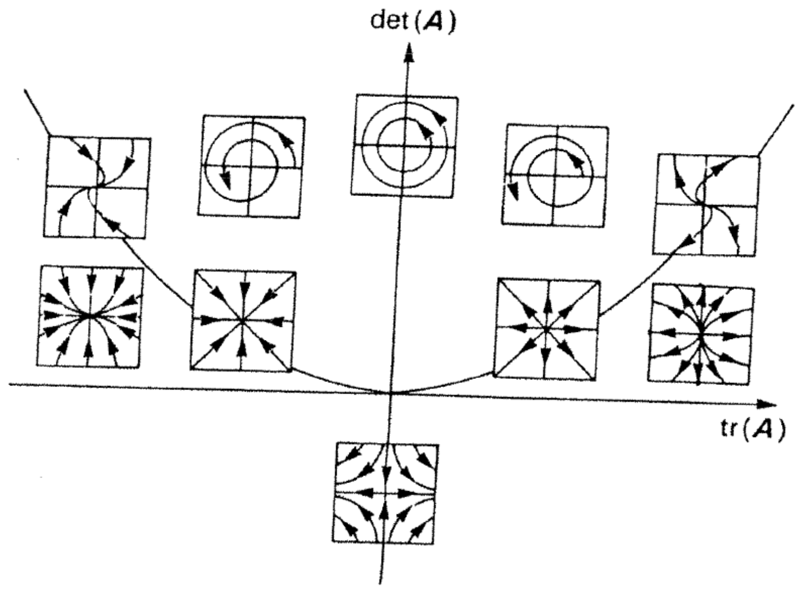
Saddle (always unstable) when eigenvalues λ_1, λ_2 are real, with different signs. $\det(A) < 0$.

Stable (unstable) focus when λ_1, λ_2 are complex, with negative (positive) real parts. $\det(A) > \frac{1}{4}(tr(A))^2 \neq 0$, $tr(A) < 0$ ($tr(A) > 0$).

Stable (unstable) improper node when eigenvalue λ_1 is real negative (positive) with multiplicity 2 having only one linearly independent eigenvector. $\det(A) = \frac{1}{4}(tr(A))^2$, $tr(A) < 0$ ($tr(A) > 0$).

Center (stable but not asymptotically stable) when λ_1, λ_2 are complex purely imaginary. $tr(A) = 0$; $\det(A) > 0$

Stable (unstable) star, when eigenvalue λ_1 is real negative (positive) with multiplicity 2 having two linearly independent eigenvectors. $\det(A) = \frac{1}{4}(tr(A))^2$, $tr(A) < 0$ ($tr(A) > 0$).



Summary of phase portraits for the system x'=Ax depending on tr(A) and det(A).
The division line is det(A)= ¼ (tr(A))².

More general conclusions about stability or instability of the equilibrium in the origin for autonomous linear systems of ODE follow immediately from the Corollary 2.13 in L.&R.

Theorem. (Propositions 5.23, 5.24, 5.25, pp. 189-190, L.R.)

Let $A \in \mathbb{C}^{N\times N}$ be a complex matrix.

Then three following statements are valid for the system $x'(t) = Ax(t)$

1. The origin is asymptotically stable if and only if $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$.

2. The origin is exponentially stable if and only if $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$.

3. The equilibrium point in the origin is stable if and only if $\operatorname{Re}\lambda \leq 0$ for all $\lambda \in \sigma(A)$ and all purely imaginary eigenvalues are semisimple.

   Definition. Matrix A with the property $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$ is called Hurwitz matrix.

1.3 Inhomogeneous linear systems with constant coefficients.

Corollary. Duhamel formula, autonomous case. (Corollary 2.17, p. 43)

Consider the inhomogeneous system

$$x'(t) = Ax + b(t)$$

with continuous or piecewise continuous function $b : \mathbb{R} \to \mathbb{R}^N$. Then the unique solution to the I.V.P. with initial data

$$x(\tau) = \xi$$

is represented by the Duhamel formula:

$$x(t) = \exp(At)\xi + \int_{\tau}^{t} \exp(A(t-\sigma))b(\sigma)d\sigma \tag{2}$$

Proof to the Corollary: check that the formula gives a solution and show that it is unique.

$$
\begin{aligned}
x(t) &= \exp(At)\xi + \int_0^t \exp(A(t-s))g(s)ds \\
&= \exp(At)\xi + \exp(At)\int_0^t \exp(-As)g(s)ds \\
&= \exp(At)\left[\xi + \int_0^t \exp(-As)g(s)ds\right] \\
x'(t) &= A\exp(At)\left[\xi + \int_0^t \exp(-As)g(s)ds\right] + \exp(At)\exp(-At)g(t) \\
&= Ax(t) + g(t)
\end{aligned}
$$

Difference $z(t) = x(t) - y(t)$ between two solutions $x(t)$ and $y(t)$ satisfies the homogeneous systems $z'(t) = Az(t)$ and zero initial condition $z(0) = 0$ and the integral equation: $z(t) = \int_0^t Az(s)ds$. The same reasoning as before, using the Grönwall inequality, implies that $z \equiv 0$.

1.4 Stability of equilibrium points to linear systems perturbed by a small right hand side.

Theorem (Theorem 5.27, p. 193, L.R.) Let $G \subset \mathbb{R}^N$ be a nonempty open subset with $0 \in G$. Consider the differential equation

$$x'(t) = Ax + h(x) \tag{3}$$

where $A \in \mathbb{R}^{N\times N}$ and $h : G \to \mathbb{R}^N$ is a continuous function satisfying

$$\lim_{z\to 0} \frac{h(z)}{\|z\|} = 0. \tag{4}$$

If A is Hurwitz, that is $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$, then 0 is an asymptotically stable equilibrium of 3.

Moreover, there is $\Delta > 0$ and $C > 0$ and $\alpha > 0$ such that for $\|\xi\| < \Delta$

the solution $x(t)$ to the initial value problem with initial data

$$x(0) = \xi$$

satisfies the estimate

$$\|x(t)\| \leq C \|\xi\| e^{-\alpha t}$$

Proof. (This proof is required at the exam)

If $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$ then there is $\beta > 0$ such that $\operatorname{Re}\lambda < -\beta$ for all $\lambda \in \sigma(A)$ and

$$\|\exp(At)\| \leq Ce^{-\beta t} \tag{5}$$

for some constant $C > 0$.

We can choose $\varepsilon > 0$ such that $C\varepsilon < \beta$ and using (4) choose δ_ε such that for $\|z\| < \delta_\varepsilon$

$$\|h(z)\| < \varepsilon \|z\| \tag{6}$$

We know that the solution to the equation (3) exists on some time interval $t \in [0, \delta)$.

We apply Duhamel formula (2) for solutions to (3):

$$x(t) = \exp(At)\xi + \int_0^t \exp(A(t-\sigma))h(x(\sigma))d\sigma$$

As long as $\|x(t)\| < \delta_\varepsilon$ we apply the triangle inequality for integrals and estimates (5) and (6):

$$\|x(t)\| \leq Ce^{-\beta t} \|\xi\| + \int_0^t Ce^{-\beta(t-\sigma)}\varepsilon \|x(\sigma)\| d\sigma$$

Introduce function $y(t) = \|x(t)\| e^{\beta t}$. Then multiplying the last inequality by $e^{\beta t}$ we arrive to

$$\|y(t)\| \le C\,\|\xi\| + \int_0^t (C\varepsilon)\, y(\sigma) d\sigma$$

The Grönwall inequality implies that

$$\|y(t)\| \le C\,\|\xi\|\, e^{(C\varepsilon)t}$$

and

$$\|x(t)\| \le C\,\|\xi\|\, e^{-(\beta - C\varepsilon)t} \tag{7}$$

It is valid as long as $\|x(t)\| < \delta_\varepsilon$. Now we can choose $\alpha = \beta - C\varepsilon > 0$, $\Delta = \frac{1}{2}\delta_\varepsilon/C$ and $\|\xi\| < \Delta$. This choice of initial conditions implies that

$$\|x(t)\| \le \delta_\varepsilon, \tag{8}$$

(Important theoretical argument!!!)

This estimate implies an important conclusion that the solution must exist in fact on the whole $\mathbb{R}_+$, because supposing opposite, namely that there is some maximal existence time $t_{\max}$ leads to a contradiction.

Let consider this important argument. It consists of two steps.

1) using the continuity and boundedness of the solution $x(t)$ on $[0, t_{\max})$ we can extend $x(t)$ up to the point $t_{\max}$ as $x(t_{\max}) = \eta = \lim_{t\to t_{\max}} x(t)$.

2) Now using the existence theorem, we conclude that there is a solution $y(t)$ to the equation

$$y'(t) = Ay + b(t)$$

on the time interval $[t_{\max}, t_{\max} + \delta)$ with the initial condition $y(t_{\max}) = \eta$ at time $t_{\max}$. This solution is evidently an extension of the original solution $x(t)$ to a larger time interval, that contradicts to ower supposition. Therefore the solution $x(t)$ can be extended to the whole $\mathbb{R}_+$ and satisfies the estimate (8).

It in turn implies that this solution must satisfy the desired estimate

$$\|x(t)\| \leq C \|\xi\| e^{-\alpha t}$$

and the asymptotic stability of the equilibrium point in the origin.■

This theorem implies immediately the following result on stability of equilibrium points by linearization.

Theorem. (Corollary 5.29, p. 195) On stability of equilibrium points by linearization.

Let $f : G \to \mathbb{R}^N$, $G \subset \mathbb{R}^N$ is non empty open set with $0 \in G$, f be continuous and $f(0) = 0.$ Let f be differentiable in 0 and A be the Jacoby matrix of f in the point 0, $a = D(f)(0)$:

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(0), \quad i, j = 1, ...N$$

If A is a Hurwitz matrix (all eigenvalues $\lambda \in \sigma(A)$ have $\mathrm{Re}\,\lambda < 0$), then the equilibrium point of the system

$$x'(t) = f(x(t))$$

in the origin is asymptotically stable.

Proof. Consider the function $h(z) = f(z) - Az$. Then by the definition of derivatives $h(z)/\|z\| \to 0$ as $z \to 0$. An application of the theorem about stability of a small perturbation of a linear system to the function $f(z) = Az + h(z)$ proves the the claim. ■

The following general theorem by Grobman and Hartman that we formulate without proof is a strong result on connection between solutions to a nonlinear system

$$\begin{aligned} x'(t) &= f(x(t)), & (9) \\ x(0) &= \xi & (10) \end{aligned}$$

with right hand side $f(x)$ close to an equilibrium point x_*, $f(x_*) = 0$ and solutions to the linearized system

$$\begin{aligned} y'(t) &= Ay && (11) \\ y(0) &= \zeta - x_* && (12) \end{aligned}$$

with constant matrix A that is Jacobi matrix of the right hand sisde f in the equilibrium point x_*, $A = D(f)(x_*)$:

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(x_*), \quad i, j = 1, ...N$$

Definition. An equilibrium point x_* of the system (9) is called hyperbolic if for all eigenvalues $\lambda \in \sigma(A)$ it is valid that $\operatorname{Re} \lambda \neq 0$.

Theorem. (Grobman-Hartman)

Let $f \in C^1(B)$, in $B_R(x_*) = \{\xi : \|\xi - x_*\| < R \quad \} \subset G$ and x_* be a hyperbolic equilibrium point of (9). Then there are neighbourhoods $U_1(x_*)$ and $U_2(x_*)$ of x_* and an invertible continuous mapping $R : U_1(x_*) \to U_2\ (x_*)$ such that R maps shifted solutions $e^{At}(\zeta - x_*)$ to the linearized system (11) onto solutions $x(t) = \varphi(t, R(\zeta))$ of the non-linear system (9) with $\xi = R(\zeta)$:

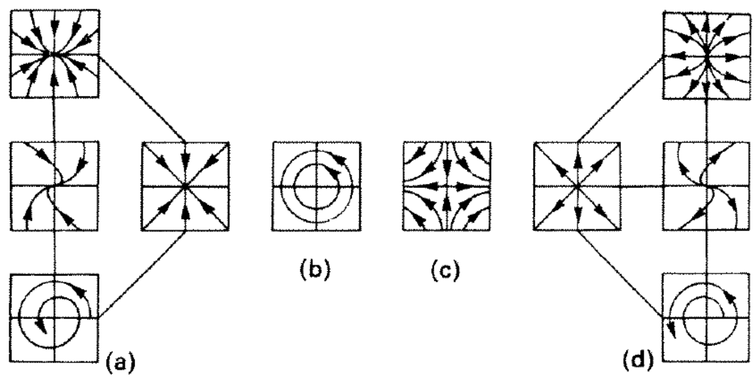
$$R\left(x_* + e^{At}(\zeta - x_*)\right) = \varphi(t, R(\zeta))$$

and back

$$R^{-1}\left(\varphi(t, \xi)\right) = x_* + e^{At}(R^{-1}(\xi) - x_*)$$

From the intuitive point of view it means that phase portraits of the non-linear and linearized systems are topologically equivalent in a neighbourhood of the hyperbolic equilibrium point x_*.

Various classes of topologically equivalent equilibrium points in the plane: a) asymptotically stable, b) center, c) saddle point, d) unstable.



Example on application of Grobman - Hartman theorem

Consider the system

$$x_1' = -\tfrac{1}{2}(x_1 + x_2) - x_1^2$$
$$x_2' = \tfrac{1}{2}(x_1 - 3x_2)$$

-0.5$(-2/3 - 2/9) - 4/9 = 0.0$

It has two equilibrium points: one in the origin $(0,0)$ and one is $(-2/3, -2/9)$.We find them by expressing $x_1 = 3x_2$, from the equation $\frac{1}{2}(x_1 - 3x_2) = 0$, substituting to the equation $-\frac{1}{2}(x_1 + x_2) - x_1^2 = 0$, and solving the quadratic equation $-\frac{1}{2}(3x_2 + x_2) - 9x_2^2 = 0$ for x_2. $-\frac{1}{2}(3x_2 + x_2) - 9x_2^2 = -x_2(9x_2 + 2) = 0$.

and its linearization in the origin:

$$x_1' = -\tfrac{1}{2}(x_1 + x_2)$$
$$x_2' = \tfrac{1}{2}(x_1 - 3x_2)$$

The linearized system has matrix $A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$, characteristic polynomial: $\lambda^2 + 2\lambda + 1 = 0$, eigenvalues: $\lambda_{1,2} = -1$. The only eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The origin is a stable improper node for both systems. This equilibrium point is asymptotically stable.

On the other hand we see that another equilibrium $(-2/3, -2/9)$ of the non-linear system seems to be a saddle point.

We check it now. For an arbitrary point we need first to calculate the Jacoby matrix of the right hand sisde in the system $x' = f(x)$ in an arbitrary point $x \in \mathbb{R}^2$

$$
\begin{aligned}
[Df]_{ij}(x) &= \frac{\partial f_i}{\partial x_j}(x) \\
[Df](x) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{bmatrix} = \begin{bmatrix} -1/2 - 2x_1 & -1/2 \\ 1/2 & -3/2 \end{bmatrix}
\end{aligned}
$$

Calculating the Jacoby matrix in the second equilibrium point $(-2/3, -2/9)$ we get the matrix for the linearization of the right hand side in this point:

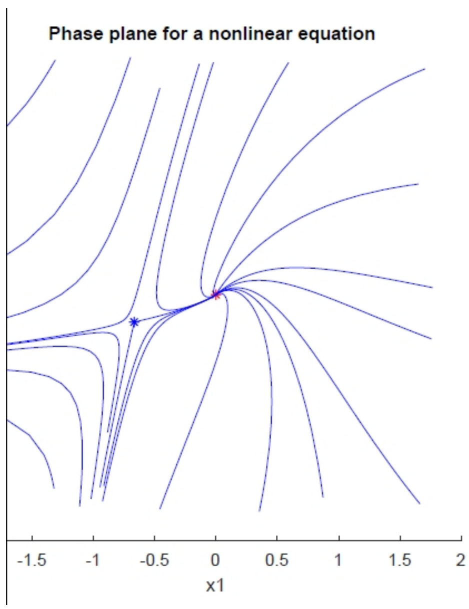
$$
A = \begin{bmatrix} -1/2 - 2(-2/3) & -1/2 \\ 1/2 & -3/2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}
$$

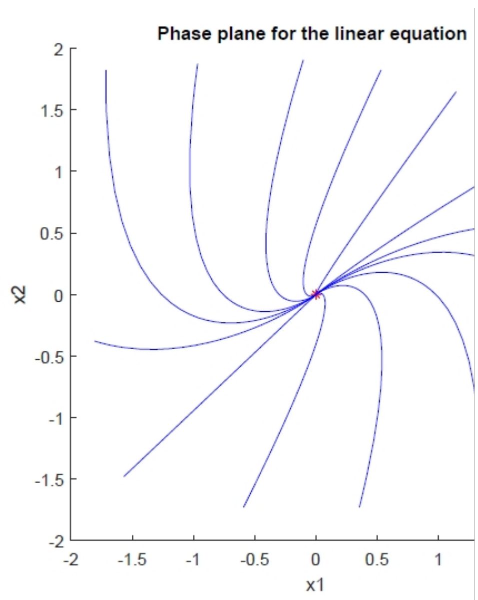
The characteristic polynomial is $p(\lambda) = \lambda^2 - \lambda \mathrm{tr}(A) + \det(A)$. $\mathrm{tr}(A) = 5/6 - 3/2 = -\frac{2}{3}$. $\det(A) = \frac{5}{6}\left(-\frac{3}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right) = -1$. Therefore $p(\lambda) = \lambda^2 + \frac{2}{3}\lambda - 1$. Eigenvalues are real and have different signs because the determinant is negative. We don to need to calculate them to make these conclusions.

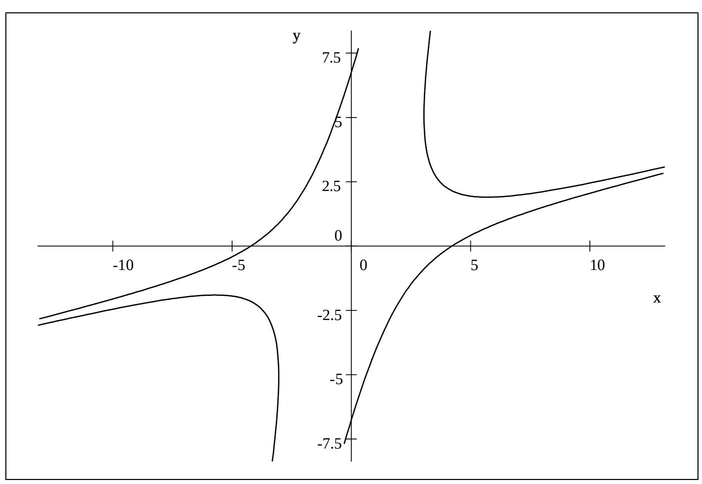
Therefore the linearized system

$$
y' = Ay
$$

has a saddle point in the origin. The non-linear system also has a saddle point configuration in the phase portrait close to the equilibrium point $(-2/3, -2/9)$ according to the Grobman-Hartman theorem. This equilibrium point is unstable. If we like to sketch a more precise phas portrait for the linearized system we can calculate eigenvalues and eigenvectors. But we can only guess the global phase protrait for the non-linear system (how local phase portraits connect with each other). We give below phase portraits for the non-linear system and for linearized systems around each of equilibrium points.







Phase plane for the linearized system around the equilibrium point $(-2/3, -2/9)$

Counterexample

A system such that the linearized system has a center (stable) but the non-linear has a non stable equilibrium point.

Cosider the system

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2 + (x_1^2 + x_2^2)x_1 \\
\frac{dx_1}{dt} &= -x_1 + (x_1^2 + x_2^2)x_2
\end{aligned}
$$

The origin $(0,0)$ is an equilibrium point and the linearized system in this point has the form

$$
x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x
$$

The origin is a center that is a stable euilibrium point.

Consider the equation for $r^2(t) = x_1^2(t) + x_2^2(t)$.We derive it by multiplying the first equation by x_1 and the second by x_2 and considering the sum of the equations leading to

$$
\frac{1}{2}\frac{d}{dt}r^2(t) = \left(r^2(t)\right)^2
$$

We see that the solution to this equation with separable variables with arbitrary initial data $r(0)$ is

$$
r^2(t) = \frac{r^2(0)}{1 - 2r^2(0)t}
$$

The solution tends to infinity with t rising and blows up in finite time.

The equilibrium $(0,0)$ to the nonlinear system is unstable. The phase portraits of the nonlinear system and the linearized system are qualitatively different in this example when eigenvalues to the Jacoby matrix of the right hand side of the nonlinear system in the equilibrium point have real parts equal to zero.