1.1.1 An example from circuit theory

The study of electrical circuits is a source of many important differential equations. Consider, for example, the circuit shown in Figure 1.1 consisting of a parallel connection of a capacitor (with capacitance C), an inductor (with inductance L), and a nonlinear resistor.

Let $J = \mathbb{R}$. At any time $t \in J$, the current $\imath_R(t)$ through the resistor is related to the voltage $v_R(t)$ across the resistor by a nonlinear function g, that is,

$$\imath_R(t) = g(v_R(t)) .$$

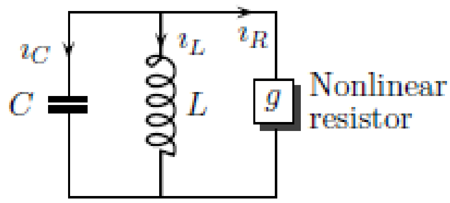

Figure 1.1

For example, if g is given by

$$g(\zeta) = -\zeta + \zeta^3, \quad \forall \zeta \in \mathbb{R}, \tag{1.1}$$

then this corresponds to a particular component known as a twin-tunnel diode.

The voltage v_L across the inductor is related to the current $\imath_L$ through the inductor by Faraday's law

$$v_L(t) = L\frac{d\imath_L}{dt}(t) , \quad L \text{ constant} .$$

The voltage v_C across the capacitor and the corresponding current $\imath_C$ satisfy the relation

$$C\frac{dv_C}{dt}(t) = \imath_C(t) , \quad C \text{ constant} .$$

Kirchoff's current and voltage laws give

$$\imath_R(t) + \imath_L(t) + \imath_C(t) = 0 \quad \forall\, t \in J$$

and

$$v_R(t) = v_L(t) = v_C(t) \quad \forall\, t \in J.$$

Eliminating the variables $\imath_C$, $\imath_R$, v_L and v_R from the above relations, yields the system of two differential equations

$$L\frac{d\imath_L}{dt}(t) = v_C(t), \qquad C\frac{dv_C}{dt}(t) = -\imath_L(t) - g(v_C(t)).$$

Defining $x_1(t) := L\imath_L(t)$ and $x_2(t) := v_C(t)$, we obtain

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = -\mu_1 x_1(t) - \mu_2 g(x_2(t)),$$

where $\mu_1 := 1/(CL)$ and $\mu_2 := 1/C$. Setting

$$f(z) = f(z_1, z_2) := \big(z_2\,,\ -\mu_1 z_1 - \mu_2 g(z_2)\big)$$

for all $z = (z_1, z_2) \in G := \mathbb{R}^2$ defines a function $f\colon G \to \mathbb{R}^2$ and, on writing $x(t) = (x_1(t), x_2(t))$, the above pair of differential equations can be written as the autonomous system $\dot{x}(t) = f(x(t))$.

Now assume, for simplicity, that $\mu_1 = \mu_2 = 1$ and consider again the case of a twin-tunnel diode described by the characteristic (1.1), in which case f is given by

$$f(z) = f(z_1, z_2) = \big(z_2\,,\ -z_1 + z_2 - z_2^3\big).$$

Note that $f(z) = 0$ if, and only if, $z = 0$. Let $\mathcal{A}$ be the "annular" region in the plane, as in Figure 1.2, wherein the inner boundary is the circle of unit radius centred at $(0,0)$ and the outer boundary is a polygon with vertices as shown. A straightforward calculation reveals that there is no point (z_1, z_2) of either the inner or the outer boundary at which the vector $f(z_1, z_2)$ is directed to the exterior of the annulus (equivalently, at each point z of each boundary, the vector $f(z)$ is either tangential or directed inwards). An immediate consequence of this observation is the following fact: if $x\colon J \to G$ is a solution on the (that is, a continuously differentiable function with $\dot{x}(t) = f(x(t))$ for all $t \in J$) with $x(0) \in \mathcal{A}$, then $x(t) \in \mathcal{A}$ for all $t \geq 0$.

The set $\mathcal{A}$ is said to be *positively invariant*: solutions starting in the set are trapped within the set in forwards time. We now know that the set $\mathcal{A}$ is positively invariant and is such that $f(z) \neq 0$ for all $z \in \mathcal{A}$. These two properties are sufficient to ensure (via the Poincaré-Bendixson theorem – to be stated and proved in Section 4.6) that the system has at least one periodic solution (that is, a solution $x \colon J \to G$ with the property that, for some $T > 0$, $x(t) = x(t+T)$ for all $t \in J$), the *orbit* of which (that is, the set $\{x(t) \colon t \in \mathbb{R}\} = \{x(t) \colon t \in [0,T)\}$) is contained in $\mathcal{A}$. The existence of a periodic orbit is reflected in the terminology "nonlinear oscillator" commonly used in the context of this circuit.

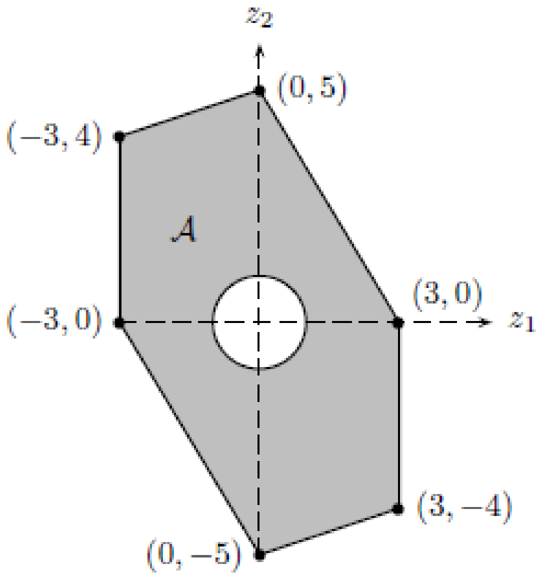

Figure 1.2 Positively invariant set $\mathcal{A}$