Extra Notes 1 (23/3)

Example E.1. For every $n \geq 0$, let q_n be the number of words in the alphabet $\mathcal{A} = \{a,\ b,\ c,\ d\}$ which have an odd number of ones.

We can compute the first couple of values directly: $q_1 = 1$,$q_2 = 6$, with the possible words given by

$$b; \quad ab, \ cb, \ db, \ ba, \ bc, \ bd.$$

I claim that, for every $n \geq 2$,

$$q_n = 2 \cdot q_{n-1} + 4^{n-1}. \tag{0.1}$$

For an admissable word of length n, consider the following two cases:

Case 1: The word begins with a b. Then the remaining letters form a word of length $n-1$ with an even number of b's. There are 4^{n-1} words of length $n-1$ in total (by MP) and q_{n-1} of them, by definition, have an odd number of b's. Hence, the number of admissable words in Case 1 is $4^{n-1} - q_{n-1}$.

Case 2: The word begins with a, c or d. Then, by a similar analysis to Case 1, there are q_{n-1} possibilities for the remaining letters. Since there are three choices for the first letter, by MP there are a total of $3 \cdot q_{n-1}$ admissable words in Case 2.

Clearly, Cases 1 and 2 are mutually exclusive and exhaust all options so, by AP, the total number of admissable words of length n is $(4^{n-1} - q_{n-1}) + 3 \cdot q_{n-1} = 2 \cdot q_{n-1} + 4^{n-1}$, which proves (0.1).

The recursion (0.1), with initial condition $q_1 = 1$ (or, if you like, $q_0 = 0$ since the empty word contains an even number of b's, namely zero of them), can be solved in the usual way using either the auxiliary equation method or generating functions. One finds that

$$q_n = \frac{1}{2}(4^n - 2^n) \quad \forall\, n \geq 0. \tag{0.2}$$

Note, in particular, that less than half of all the 4^n words of length n have an odd number of b's, though the proportion approaches one half as $n \to \infty$, as one would expect.