Extra Notes 2 (16/4)

It is a well-known fact (see any textbook on Group Theory) that every permutation of a finite set has a unique representation as a product of disjoint cycles, that is, as a composition of cyclic permutations which involve pairwise disjoint subsets of the set and hence commute.

For $n \in \mathbb{N}$ and a subset $\{x_1, x_2, \ldots, x_k\} \subseteq [n]$, we will use the notation

$$(x_1\, x_2\, \ldots\, x_k) \tag{0.3}$$

to denote the cyclic permutation $\pi : [n] \to [n]$ such that

$$\pi(x_1) = x_2, \;\; \pi(x_2) = x_3, \;\ldots,\; \pi(x_k) = x_1, \;\; \pi(y) = y \;\forall\, y \in [n] \backslash \{x_1, \ldots, x_k\}. \tag{0.4}$$

Remark E.2. This is one of two conventions common in the literature. Sometimes the notation (0.3) is used to denote the inverse of the permutation in (0.4).

Example E.3. Let $\pi : [8] \to [8]$ be given by

i	1	2	3	4	5	6	7	8
$\pi(i)$	4	1	7	2	8	5	3	6

Then, in cycle notation,

$$\pi = (1\,4\,2)(3\,7)(5\,8\,6).$$

Definition E.4. Let k, n be non-negative integers with $k \leq n$. The *Stirling number of the first kind* $s(n, k)$ is the number of permutations of $[n]$ consisting of exactly k disjoint cycles.

Remark E.5. Once again, there are other conventions in the literature regarding the definition of Stirling numbers of the first kind. Check the `Wikipedia` entry, for example.

Theorem E.6. *With the definition as in E.4 above, we have the recurrence*

$$s(n, n) = 1 \;\forall\, n \geq 0; \quad s(n, 0) = s(0, n) = 0 \;\forall\, n \geq 1; \tag{0.5}$$

$$s(n+1, k) = n \cdot s(n, k) + s(n, k-1) \;\forall\, n \geq 0, \; 1 \leq k \leq n. \tag{0.6}$$

Proof: Eqs. (0.5) are obvious: note that, if $n \geq 1$, the only permutation of $[n]$ consisting of n cycles is the identity permutation. So we turn to (0.6). Let π be a permutation of $[n+1]$ containing k cycles. We consider two cases:

CASE 1: $n+1$ forms a cycle on its own. In other words, $\pi(n+1) = n+1$. Then the restriction of π to $[n]$ is a permutation of the latter involving $k-1$ cycles. So the number of possibilities for π in Case 1 is $s(n, k-1)$.

CASE 2: $\pi(n+1) = j$, for some $j \in [n]$. Let π^* be the following permutation of $[n]$:

$$\pi^*(i) = \begin{cases} \pi(i), & \text{if } \pi(i) \neq n+1, \\ j, & \text{if } \pi(i) = n+1. \end{cases}$$

We consider in turn two subcases:

Subcase 2.A: j forms a cycle on its own in π^*. This means that $\pi(j) = n+1$ and hence one of the cycles in π is the involution $(j\ n+1)$. It is easy to see that all other cycles are the same in π and π^*. Hence, if π has k cycles then so does π^*.

Subcase 2.B: j is part of a cycle of length at least 2 in π^*. Say that the cycle is

$$(j\, x_1\, x_2\ \dots\ x_r).$$

In terms of π this means that

$$\pi(j) = x_1,\ \ \pi(x_1) = x_2,\ \dots,\ \pi(x_{r-1}) = x_r,\ \ \pi(x_r) = n+1.$$

Together with the fact that $\pi(n+1) = j$, this means that π contains the cycle

$$(n+1\ j\ x_1\ \dots\ x_r).$$

It is also easy to see in this case that all remaining cycles of π and π^* coincide so, once again, if π has k cycles then so does π^*.

To summarise, the map $\pi \to \pi^*$ establishes a 1-1 correspondence between the permutations of $[n+1]$ which involve k cycles and send $n+1$ to some fixed element of $[n]$ and *all* permutations of $[n]$ involving k cycles. Hence, given $j \in [n]$, there are $s(n,\, k)$ possibilities for π. Since there are n choices for j, the total number of possible permutations in Case 2 is $n \cdot s(n,\, k)$.

A final application of the addition principle then yields (0.6).