Extra Notes 3 (25/4)

Definition E.7. Let G be a (simple) graph containing at least one edge. For each $n \geq |V(G)|$ we denote by ex$(n,\, G)$ the largest integer $e = e_n$ such that there exists a simple graph with n vertices and e edges containing no subgraph isomorphic to G.

Note that ex$(n,\, G) < \binom{n}{2}$ since K_n contains a copy of every graph on at most n vertices. We study the case when $G = K_k$, a complete graph. It is obvious that ex$(n,\, K_2) = 0$. Already the next step is non-trivial.

Lemma E.8. (Cauchy-Schwarz inequality) *For any $n \in \mathbb{N}$ and any two vectors $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$ one has*

$$|\boldsymbol{x} \cdot \boldsymbol{y}|^2 \leq ||\boldsymbol{x}||^2 ||\boldsymbol{y}||^2. \tag{0.7}$$

Proof: For any $t \in \mathbb{R}$ one has

$$0 \leq ||\boldsymbol{x} - t\boldsymbol{y}||^2 = (\boldsymbol{x} - t\boldsymbol{y}) \cdot (\boldsymbol{x} - t\boldsymbol{y}) =$$
$$= (\boldsymbol{x} \cdot \boldsymbol{x}) - 2t(\boldsymbol{x} \cdot \boldsymbol{y}) + t^2(\boldsymbol{y} \cdot \boldsymbol{y}) = ||\boldsymbol{x}||^2 - 2t(\boldsymbol{x} \cdot \boldsymbol{y}) + t^2||\boldsymbol{y}||^2.$$

The RHS is thus a positive semi-definite quadratic function of t, which means that its discriminant must be non-positive, i.e.:

$$b^2 \leq 4ac \Leftrightarrow |\boldsymbol{x} \cdot \boldsymbol{y}|^2 \leq ||\boldsymbol{x}||^2 ||\boldsymbol{y}||^2, \quad \text{v.s.v.}$$

Corollary E.9. *For any real numbers $x_1,\, x_2,\, \ldots,\, x_n$ one has*

$$\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2. \tag{0.8}$$

Proof: Apply (0.7) to the pair $\boldsymbol{x} = (x_1,\, x_2,\, \ldots,\, x_n)$, $\boldsymbol{y} = (1,\, 1,\, \ldots,\, 1)$.

Theorem E.10. *ex$(n,\, K_3) = \lfloor n^2/4 \rfloor$.*

Proof: The complete bipartite graph $K_{\lfloor n/2 \rfloor,\, \lceil n/2 \rceil}$ contains exactly $\lfloor n^2/4 \rfloor$ edges and no K_3. Now let $G = (V,\, E)$ be any graph on n vertices not containing any K_3. We must show that $|E| \leq n^2/4$. Consider

$$\sum_{\{v,\, w\} \in E} \text{deg}(v) + \text{deg}(w). \tag{0.9}$$

On the one hand, if for some edge $\{v,\, w\}$ one had deg(v)+ deg$(w) > n$ then, by the Pigeonhole Principle, there would have to exist a third vertex $x \in V$ which is a common neighbor of v and w. In that case, $\{v,\, w,\, x\}$ would form a K_3 in G. Hence, each term in (0.9) is at most n and so

$$\sum_{\{v,\, w\} \in E} \text{deg}(v) + \text{deg}(w) \leq n \cdot |E|. \tag{0.10}$$

On the other hand, for each $v \in V$, deg(v) appears in exactly deg(v) terms of the sum (0.9). In other words, the sum is identical to $\sum_{v \in V} (\text{deg}(v))^2$. By Corollary E.9 and

Theorem 13.12,

$$\sum_{v\in V}(\deg(v))^2 \geq \frac{1}{|V|}\left[\sum_{v\in V}\deg(v)\right]^2 = \frac{1}{n}(2|E|)^2 = \frac{4|E|^2}{n}.$$

Thus,

$$\sum_{\{v,\,w\}\in E} \deg(v)+\deg(w) \geq \frac{4|E|^2}{n}. \tag{0.11}$$

From (0.10) and (0.11) it follows that

$$\frac{4|E|^2}{n} \leq n\cdot|E| \Rightarrow \cdots \Rightarrow |E| \leq \frac{n^2}{4}, \quad \text{v.s.v.}$$

Definition E.11. Let $r \geq 2$. A graph $G = (V,\, E)$ is said to *r-partite* if there is a partition

$$V = \bigsqcup_{i=1}^{r} V_i, \quad V_i \neq \phi, \tag{0.12}$$

such that no edge in G is between a pair of vertices in the same V_i.

Let $n_1,\, \ldots,\, n_r$ be positive integers. The *complete r-partite graph* $K_{n_1,\ldots,n_r} = (V,\, E)$ satisfies, in the notation of (0.12), $|V_i| = n_i$ for each i and $E = \{\{v_i,\, v_j\} : v_i \in V_i,\, v_j \in V_j,\, i \neq j\}$.

Theorem E.10 can be generalised in the following way. Let $r \geq 2$ and $n \geq r$. We can uniquely write $n = qr + t$, where $0 \leq t < r$. Set $n_1 = \cdots = n_t = q+1$, $n_{t+1} = \cdots = n_r = q$. Then

Theorem E.11. (Turán's Theorem) *ex*$(n,\, K_{r+1}) = |E(K_{n_1,\ldots,n_r})|$.

The proof of this result is beyond the scope of our course, but is easy to locate in the literature.

Remark E.12. The proof of Theorem E.10 employed the fact that a bipartite graph contains no K_3. More generally, a bipartite graph contains no odd cycles (see Theorem 16.9). So if a graph G contains even cycles, we can't employ bipartite graphs as freely when attempting to construct dense graphs without any copies of G. This turns out to make a big difference. For example, it is known that $\text{ex}(n,\, C_4) = O(n^{3/2})$. For a fairly recent discussion of such matters, see for example

`https://www.sciencedirect.com/science/article/pii/S0095895613000038`.