First Exercise Session: 9/4

Theme: Combinatorics

Relevant Chapters: 6, 10, 11

1. (classic) (a) Let p be a prime. Prove that, if x and y are integers such that $x^2 \equiv y^2 \pmod{p}$, then either $x \equiv y \pmod{p}$ or $x \equiv -y \pmod{p}$.

(b) With the help of the pigeonhole principle deduce that, for any prime p, the congruence $a^2 + b^2 + 1 \equiv 0 \pmod{p}$ has a solution.

2. (5.57 in EG-1) A certain obsessive-compulsive person always pairs off their socks when they take them out of the washing machine.

(a) If there are 20 socks in the machine, how many must he, in the worst case, take out before he can be certain of having at least one pair ?

(b) What is the probability that he will indeed remove so many socks before finding a pair, if he just fishes blindly in the machine ?

3. (2004 Hw 1.3) The surface of a football consists of a network of 12 pentagons and 20 hexagons. At each "node" two hexagons and a pentagon meet. Use this information to compute the number of nodes.

4. (5.35 and 5.36 in EG-1) (a) Twenty people stand in a supermarket queue. A new checkout opens and 8 people rush to it. In how many ways can this new queue of 8 people be built ?

(b) Twenty in a queue as before. This time, two new checkouts open simoultaneously and 8 people rush to them. In how many ways can the two new queues be built, if at least two people must end up in each of them ?

5. (6.17 in JL) Six people are to be chosen from a group of 7 boys and 5 girls. In how many ways can this be done if

(a) one can choose freely ?
(b) one must choose three of each sex ?
(c) one must choose at least two of each sex ?
(d) one can choose freely except that one mustn't choose both Alice and Bob, who don't get along ?

6. (6.15 in JL) (a) How many words can be made from the letters in JONASSON ?

(b) In how many of these words does the same letter never appear twice in succession ?

7. (5.65 in EG-1) We have 5 mathematicians and 20 cakes.

(a) In how many ways can the cakes be distributed among the mathematicians ?
(b) Same as above, except that each person must get at least two cakes ?
(c) Same as above, except that not all the cakes need to be eaten ?