Exercise Session 1 (9/4): Solutions

1. (a) All congruences are mod p, where p is a prime. If $x^2 \equiv y^2$ then $p \mid x^2 - y^2$, so $p \mid (x-y)(x+y)$ and, since p is prime, it follows from the Fundamental Theorem of Arithmetic[1] that either $p \mid x - y$ or $p \mid x + y$. In the former case, $x \equiv y$ and in the latter case, $x \equiv -y$.

(b) Equivalently, we must show there exist elements $a,\ b$ of $\mathbb{Z}_p$ such that $a^2 = -1 - b^2$. Now, if $x,\ y \in \mathbb{Z}_p$ then it follows from part (a) that $x^2 = y^2 \Leftrightarrow x = \pm y$. Thus each non-zero element of $\mathbb{Z}_p$ has either zero or two square roots, while 0 has only itself as a square root. In other words, as x ranges over all p elements of $\mathbb{Z}_p$, x^2 attains $1 + \frac{1}{2}(p-1) = \frac{p+1}{2}$ different values. The same is true for the expression $-1 - x^2$, since neither reflection in the origin nor translation affect the size of the image.

Hence, when we consider the congruence

$$a^2 \equiv -1 - b^2 \ (\text{mod}\ p),$$

there are exactly $\frac{p+1}{2}$ possibilities for both the left- and the right-hand side (mod p). But $\frac{p+1}{2} + \frac{p+1}{2} > p$ so, by the Pigeonhole Pirnicple, some congruence class must be attained on both sides. In other words, there do indeed exist integers $a,\ b$ such that $a^2 \equiv -1 - b^2$ (mod p).

2. (a) There are 20 "pigeonholes", one for each pair of socks. Once he has at least 21 socks (i.e.: "pigeons"), then at least two must go in the same pigeonhole (i.e.: be a pair).

ANSWER: 21.

(b) He'll have to wait until the 21st sock if and only if the first 20 are all in different pairs. The probability is A/B, where B is the total number of possibilities for a collection of 20 socks, and A is the number of such collections which contain no pairs. We have $B = \binom{40}{20}$ and $A = 2^{20}$, the latter since there are 2 possible socks to choose from in each pair.

ANSWER: $2^{20}/\binom{40}{20}$.

3. We count the set of all pairs $(v,\ r)$, where v is a node and r is a region (i.e.: a pentagon or a hexagon) to which v is incident. We are told that there are three regions r incident to each v, hence the number of pairs is $3V$, where V is the number of nodes. On the other hand, each of the 12 pentagons has 5 nodes and each of the 20 hexagons has 6 nodes, to the total number of node-region pairs must be $5 \times 12 + 6 \times 20 = 180$. Thus $3V = 180$, so $V = 60$.

4. (a) $P(20,\ 8) = 5,079,110,400$.

[1] Actually, it follows from Euclid's Lemma that, if p is prime and $a,\ b$ are integers such that $p \mid ab$, then p must divide at least one of a and b. This is in fact a step in the proof of FTA, rather than a consequence of it.

(b) Let the two new checkouts be C_1 and C_2 (they are obviously distinguishable), and let c_1, c_2 be the number of people in each. We're told that $c_1 + c_2 = 8$ and each $c_i \geq 2$. This leaves five possibilities:

$$(c_1, c_2) \in \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}.$$

No matter what the pair (c_1, c_2) is, we can imagine filling both queues at once by first choosing eight people in order, in $P(20, 8)$ ways, and then placing the first c_1 people in the first queue and the remaining c_2 in the second queue. This means that the total number of possibilities for the pair of queues is $5 \times P(20, 8) = 25,395,552,000$.

5. (a) $\binom{12}{6} = 924$.
(b) $\binom{7}{3} \times \binom{5}{3} = 35 \times 10 = 350$.
(c) $\binom{7}{2}\binom{5}{4} + \binom{7}{3}\binom{5}{3} + \binom{7}{4}\binom{5}{2} = 21 \times 5 + 35 \times 10 + 35 \times 10 = 805$.
(d) If both Pelle and Anna are chosen, then it remains to choose 4 people from 10, which can be done in $\binom{10}{4} = 210$ ways. Thus the number of ways to choose the group which avoids this problem is $924 - 210 = 714$.

6. (a) $\frac{8!}{(2!)^3} = 5040$.

(b) Let $\mathcal{X}$ denote the set of all possible words and let $\mathcal{S}$, $\mathcal{O}$, $\mathcal{N}$ be the subsets consisting of those words in which SS, OO and NN occur respectively. We seek $|\mathcal{X} \backslash (\mathcal{S} \cup \mathcal{O} \cup \mathcal{N})|$. By the Inclusion-Exclusion principle,

$$\begin{aligned} |\mathcal{X} \backslash (\mathcal{S} \cup \mathcal{O} \cup \mathcal{N})| &= |\mathcal{X}| - |\mathcal{S}| - |\mathcal{O}| - |\mathcal{N}| \\ &+ |\mathcal{S} \cap \mathcal{O}| + |\mathcal{S} \cap \mathcal{N}| + |\mathcal{O} \cap \mathcal{N}| - |\mathcal{S} \cap \mathcal{O} \cap \mathcal{N}|. \end{aligned} \tag{0.1}$$

In part (a) we have already computed $|\mathcal{X}| = 5040$.

Next consider $|\mathcal{S}|$, say. If the two S's occur together, then we can imagine that we have a total of 7 letters instead of 8, namely: J, O, N, A, SS, O, N. The number of possible words is then $\frac{7!}{(2!)^2} = 1260$. Thus, $|\mathcal{S}| = |\mathcal{O}| = |\mathcal{N}| = 1260$.

Next consider $|\mathcal{S} \cap \mathcal{O}|$, say. If the two S's occur together and also the two O's, then we can imagine that we have a total of 6 letters instead of 8, namely: J, OO, N, A, SS, N. The number of possible words is then $\frac{6!}{2!} = 360$. Thus, $|\mathcal{S} \cap \mathcal{O}| = |\mathcal{S} \cap \mathcal{N}| = |\mathcal{O} \cap \mathcal{N}| = 360$.

Finally consider $|\mathcal{S} \cap \mathcal{O} \cap \mathcal{N}|$. If the two S's occur together, as well as the two O's, and also the two N's, then we can imagine that we have a total of 5 letters instead of 8, namely: J, OO, NN, A, SS. The number of possible words is then $5! = 120$. Thus, $|\mathcal{S} \cap \mathcal{O} \cap \mathcal{N}| = 120$.

Substituting everything into (0.1), we obtain

$$|\mathcal{X} \backslash (\mathcal{S} \cup \mathcal{O} \cup \mathcal{N})| = 5040 - 3 \times 1260 + 3 \times 360 - 120 = 2220.$$

7. (a) Let x_i be the number of cakes eaten by mathematician number i. Then we seek the number of solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20, \quad x_i \in \mathbb{N}_0,$$

which is $\binom{20+5-1}{5-1} = \binom{24}{4} = 10626$.

(b) Now $x_i \geq 2$ for every i. Let $y_i := x_i - 2$. Then we seek the number of solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 = 10, \quad y_i \in \mathbb{N}_0,$$

which is $\binom{10+5-1}{5-1} = \binom{14}{4} = 1001$.

(c) Let x_6 be the number of uneaten cakes. Then we seek the number of solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20, \quad x_i \in \mathbb{N}_0,$$

which is $\binom{20+6-1}{6-1} = \binom{25}{5} = 53130$.