Second Exercise Session: 16/4

Theme: Recursion

Relevant Chapters: 19, 25, 12

1. (4.6(a) in EG-2) Solve the recursion

$$a_1 = 4, \quad a_n - 3a_{n-1} = 2n - 7 \ \ \forall\, n > 1,$$

on the one hand using the auxiliary equation method and, on the other, using generating functions.

2. (4.6(b) in EG-2) Solve the recursion

$$b_0 = 3, \ \ b_1 = 9, \quad b_{n+2} - 5b_{n+1} + 6b_n = 2 \cdot 4^n \ \ \forall\, n \geq 0,$$

on the oe hand using the auxiliary equation method and, on the other, using generating functions.

3. (4.8 in EG-2) Derive and solve a recursion for the number of n-digit positive integers which have an odd number of ones. What proportion of all n-digit numbers do these comprise ?

4. (6.19 in EG-2) Let $A(x)$ and $B(x)$ be the generating functions for the sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ respectively.

(a) For which sequence is $A(x) + B(x)$ the generating function ?
(b) For which sequence is $A(x)B(x)$ the generating function ?
(c) For which sequence is $A(x^2)$ the generating function ?
(d) For which sequence is the derivative of $A(x)$ the generating function ?
(e) For which sequence is $(A(x) - a_0)/x$ the generating function ?
(f) Let $a_{-1} \in \mathbb{R}$. For which sequence is $xA(x) + a_{-1}$ the generating function ?

5. Determine, with proof, the relationship between the following sequences and the Catalan numbers:

(a) For $n \geq 1$, a_n is the number of ways of computing a product $\prod_{i=1}^{n} x_i$ of non-commuting terms as a sequence of $n - 1$ multiplications.
(b) For $n \geq 1$, b_n is the number of ways of placing $2n$ points round the circumference of a circle and drawing n chords between pairs in such a way that no two chords intersect.

6. Compute the Stirling number $S(5,\, 3)$ and verify the answer by writing down all possible ways of partitioning the set $\{1,\, 2,\, 3,\, 4,\, 5\}$ into three parts.

7. Let $n = \prod_{i=1}^{k} p_i^{e_i}$ be a natural number with given prime factorisation. A factorisation $n = n_1 n_2$ is said to be *non-trivial* if $\min\{n_1, n_2\} > 1$.

(a) Determine the number of non-trivial factorisations $n = n_1 n_2$ as a Stirling number, when n is *squarefree*, i.e.: $e_i = 1$ for every i.

(b) Determine a formula for the number of non-trivial factorisations $n = n_1 n_2$ in general.