Exercise Session 2 (16/4): Solutions

1. *Auxiliary equation method:* The auxiliary equation is $x-3=0$, so the homogeneous part of the solution is $a_{h,n} = C_1 \cdot 3^n$. Our guess for the particular solution is $a_{p,n} = C_2 \cdot n + C_3$. Insertion into the recurrence gives

$$\begin{aligned} a_{p,n} - 3a_{p,n-1} = 2n-7 \Rightarrow [C_2 n + C_3] - 3[C_2(n-1)+C_3] &= 2n-7 \\ \Rightarrow n(-2C_2) + (3C_2 - 2C_3) &= 2n-7 \\ \Rightarrow -2C_2 = 2 \text{ and } 3C_2 - 2C_3 &= -7 \\ \Rightarrow C_2 = -1, \quad C_3 &= 2. \end{aligned}$$

Hence,

$$a_n = a_{h,n} + a_{p,n} = C_1 \cdot 3^n - n + 2.$$

Inserting the initial condition yields

$$a_1 = 4 = 3C_1 - 1 + 2 \Rightarrow C_1 = 1.$$

So, finally,

$$a_n = 3^n - n + 2. \tag{0.1}$$

Generating function method: To simplify notation a bit, set $u_n := a_{n+1}$ so that the recursion in terms of u_n reads

$$u_0 = 4, \quad u_{n-1} - 3u_{n-2} = 2n-7 \ \forall n > 1,$$

which is in turn equivalent to

$$u_0 = 4, \quad u_{n+1} - 3u_n = 2(n+2) - 7 = 2n - 3 \ \forall n \geq 0.$$

Let $U(x) := \sum_{n=0}^{\infty} u_n x^n$. Given that $u_0 = 4$ and the recursion we can write

$$\begin{aligned} U(x) = 4 + \sum_{n=1}^{\infty} u_n x^n = 4 + x\left(\sum_{n=0}^{\infty} u_{n+1} x^n\right) &= \\ = 4 + x\left(3\sum_{n=0}^{\infty} u_n x^n + 2\sum_{n=0}^{\infty} n x^n - 3\sum_{n=0}^{\infty} x^n\right)&. \end{aligned} \tag{0.2}$$

The last two sums on the RHS of (0.2) come from the inhomogeneity, so let's first concentrate on how to handle these, i.e.: on how to express them as rational functions. The last sum is just the geometric series in eq. (5.4) from the lecture notes:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \tag{0.3}$$

The second-last sum is handled by differentiating both sides of this, as in Example 7.2 from the notes, which yields

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}. \tag{0.4}$$

We now substitute (0.3) and (0.4) into (0.2) and continue:

$$U(x) = 4 + x\left(3U(x) + \frac{2x}{(1-x)^2} - \frac{3}{1-x}\right)$$

$$\Rightarrow (1-3x)U(x) = 4 + \frac{2x^2}{(1-x)^2} - \frac{3x}{1-x} =$$

$$= \frac{4(1-x)^2 + 2x^2 - 3x(1-x)}{(1-x)^2} = \cdots = \frac{9x^2 - 11x + 4}{(1-x)^2}$$

$$\Rightarrow U(x) = \frac{9x^2 - 11x + 4}{(1-3x)(1-x)^2}.$$

Since the denominator of the rational function has a repeated factor, the partial fraction decomposition takes the form

$$\frac{9x^2 - 11x + 4}{(1-3x)(1-x)^2} = \frac{A}{1-3x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

$$\Rightarrow 9x^2 - 11x + 4 = A(1-x)^2 + B(1-3x)(1-x) + C(1-3x)$$

$$= (A+B+C) + x(-2A - 4B - 3C) + x^2(A + 3B)$$

$$\Rightarrow A + B + C = 4, \quad 2A + 4B + 3C = 11, \quad A + 3B = 9.$$

Set up the augmented matrix and perform Gauss elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 4 & 3 & 11 \\ 1 & 3 & 0 & 9 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & -1 & 5 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & -2 & 2 \end{array}\right].$$

Back substitution gives $C = -1$, $B = 2$, $A = 3$. Hence,

$$U(x) = \frac{3}{1-3x} + \frac{2}{1-x} - \frac{1}{(1-x)^2}.$$

Now we are ready to apply the Binomial Theorem:

$$U(x) = 3\sum_{k=0}^{\infty}(3x)^k + 2\sum_{k=0}^{\infty} x^k - \sum_{k=0}^{\infty}(k+1)x^k.$$

Comparing coefficients of x^n, we get

$$u_n = 3 \cdot 3^n + 2 - (n+1) = 3^{n+1} - n + 1.$$

Finally, then,

$$a_n = u_{n-1} = 3^{(n-1)+1} - (n-1) + 1 = 3^n - n + 2,$$

in accordance with (0.1).

2. *Auxiliary equation method:* The auxiliary equation is $x^2 - 5x + 6 = 0$, with roots $x_1 = 2$, $x_2 = 3$. So $b_{h,n} = C_1 \cdot 2^n + C_2 \cdot 3^n$. Our guess for the particular solution is $b_{p,n} = C_3 \cdot 4^n$. Inserting into the recurrence

$$b_{n+2} - 5b_{n+1} + 6b_n = 2 \cdot 4^n \Rightarrow C_3 \cdot 4^{n+2} - 5C_3 \cdot 4^{n+1} + 6C_3 \cdot 4^n = 2 \cdot 4^n$$

$$\overset{\text{divide by } 4^n}{\Rightarrow} 16C_3 - 20C_3 + 6C_3 = 2 \Rightarrow C_3 = 1.$$

Hence,

$$b_n = b_{h,n} + b_{p,n} = C_1 \cdot 2^n + C_2 \cdot 3^n + 4^n.$$

Inserting the initial conditions

$$\begin{aligned} n = 0: \quad & b_0 = 3 = C_1 + C_2 + 1 \Rightarrow C_1 + C_2 = 2, \\ n = 1: \quad & b_1 = 9 = 2C_1 + 3C_2 + 4 \Rightarrow 2C_1 + 3C_2 = 5. \end{aligned}$$

The solution is $C_1 = C_2 = 1$. Hence, finally,

$$b_n = 2^n + 3^n + 4^n. \tag{0.5}$$

Generating function method: Let $B(x) := \sum_{n=0}^{\infty} b_n x^n$. Given that $b_0 = 3$ we can write, firstly,

$$\begin{aligned} B(x) = 3 + \sum_{n=1}^{\infty} b_n x^n = 3 + x \left(\sum_{n=0}^{\infty} b_{n+1}\, x^n \right) \Rightarrow \\ \Rightarrow \sum_{n=0}^{\infty} b_{n+1}\, x^n = \frac{B(x) - 3}{x}. \end{aligned} \tag{0.6}$$

Then, using also $b_1 = 9$ and the recursion,

$$\begin{aligned} B(x) &= 3 + 9x + \sum_{n=2}^{\infty} b_n x^n = (3 + 9x) + x^2 \left(\sum_{n=0}^{\infty} b_{n+2}\, x^n \right) = \\ &= (3 + 9x) + x^2 \left(5 \sum_{n=0}^{\infty} b_{n+1} x^n - 6 \sum_{n=0}^{\infty} b_n x^n + 2 \sum_{n=0}^{\infty} 4^n x^n \right). \end{aligned} \tag{0.7}$$

The last sum on the RHS of (0.7) comes from the inhomogeneity, so let's first concentrate on how to handle this, i.e.: on how to express it as a rational function. Indeed it is just a geometric series:

$$\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n = \frac{1}{1 - 4x}. \tag{0.8}$$

We now substitute (0.8) and (0.6) into (0.7) and continue:

$$\begin{aligned} B(x) &= (3 + 9x) + x^2 \left(5 \times \frac{B(x) - 3}{x} - 6B(x) + \frac{2}{1 - 4x} \right) \\ \Rightarrow B(x) &= (3 + 9x) + 5x(B(x) - 3) - 6x^2 B(x) + \frac{2x^2}{1 - 4x} \\ \Rightarrow (1 - 5x + 6x^2)B(x) &= 3 - 6x + \frac{2x^2}{1 - 4x} = \frac{3 - 18x + 26x^2}{1 - 4x} = (1 - 2x)(1 - 3x)B(x) \\ \Rightarrow B(x) &= \frac{3 - 18x + 26x^2}{(1 - 2x)(1 - 3x)(1 - 4x)}. \end{aligned}$$

The partial fraction decomposition takes the form

$$\frac{3-18x+26x^2}{(1-2x)(1-3x)(1-4x)} = \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{1-4x}$$

$$\Rightarrow 3-18x+26x^2 = A(1-3x)(1-4x) + B(1-2x)(1-4x) + C(1-2x)(1-3x)$$

$$\Rightarrow 3-18x+26x^2 = (A+B+C) + x(-7A-6B-5C) + x^2(12A+8B+6C)$$

$$\Rightarrow A+B+C=3, \quad 7A+6B+5C=18, \quad 12A+8B+6C=26.$$

Set up the augmented matrix and perform Gauss elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 7 & 6 & 5 & 18 \\ 12 & 8 & 6 & 26 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -2 & -3 \\ 0 & -4 & -6 & -10 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{array}\right].$$

Back substitution gives $C=1$, $B=1$, $A=1$. Hence,

$$B(x) = \frac{1}{1-2x} + \frac{1}{1-3x} + \frac{1}{1-4x}.$$

Now we are ready to apply the Binomial Theorem:

$$B(x) = \sum_{k=0}^{\infty}(2x)^k + \sum_{k=0}^{\infty}(3x)^k + \sum_{k=0}^{\infty}(4x)^k.$$

Comparing coefficients of x^n, we get

$$b_n = 2^n + 3^n + 4^n,$$

in agreement with (0.5).

3. Let a_n denote the number of such n-digit numbers. We have $a_1 = 1$, since obviously the only single-digit number with an odd number of ones is 1. I claim that, for all $n \geq 1$,

$$a_{n+1} = 8a_n + 9 \cdot 10^{n-1}. \tag{0.9}$$

To see this, consider $(n+1)$-digit numbers satisfying our requirement and the following two cases:

CASE 1: The last digit is 1. In this case, the first n digits comprise an n-digit number with an even number of ones. So the number of possibilities equals the number of n-digit numbers *not* satisfying our requirement. This is $9\cdot 10^{n-1} - a_n$, since $9 \cdot 10^{n-1}$ is the total number of n-digit numbers (there are 10 choices for each digit except the first, which can't be a zero).

CASE 2: The last digit is not 1. Then there are 9 possibilities for the last digit. The first n digits comprise a number with an odd number of ones, so there are a_n possibilities for it. So we have a total of $9a_n$ possibilities in this case.

From Cases 1 and 2 together it follows that $a_{n+1} = (9\cdot 10^{n-1} - a_n) + 9a_n = 8a_n + 9 \cdot 10^{n-1}$, as claimed.

We now solve the recurrence using the auxiliary equation. This is $x - 8 = 0$, so $a_{h,\,n} = C_1 \cdot 8^n$. We have $a_{p,\,n} = C_2 \cdot 10^n$. Inserting into (0.9),

$$C_2 \cdot 10^{n+1} = 8C_2 \cdot 10^n + 9 \cdot 10^{n-1} \Rightarrow \text{ (divide by } 10^{n-1}) \\ \Rightarrow 100C_2 = 80C_2 + 9 \Rightarrow C_2 = \frac{9}{20}.$$

Hence, $u_n = u_{h,\,n} + u_{p,\,n} = C_1 \cdot 8^n + \frac{9}{20} \cdot 10^n$. Inserting the initial condition gives

$$a_1 = 1 = 8C_1 + \frac{9}{20} \cdot 10 \Rightarrow C_1 = -\frac{7}{16}.$$

Thus,

$$a_n = \frac{9}{20} \cdot 10^n - \frac{7}{16} \cdot 8^n = \frac{1}{2}\left(9 \cdot 10^{n-1} - 7 \cdot 8^{n-1}\right).$$

Remark: Note that a_n is a bit less than half of $9 \cdot 10^{n-1}$. In other words, a bit less than half of all n-digit numbers have an odd number of ones.

4. Let $(c_n)_{n=0}^{\infty}$ denote the sequence whose generating function is being asked about. We are supposed to express c_n in terms of the a:s and b:s.

(a) $c_n = a_n + b_n$.
(b) $c_n = \sum_{m=0}^{n} a_m b_{n-m}$.
(c) $c_n = a_{n/2}$ for n even and $c_n = 0$ for n odd.
(d) $c_n = (n+1)a_{n+1}$.
(e) $c_n = a_{n+1}$.
(f) $c_n = a_{n-1}$.

5. (i) I claim that $a_n = C_{n-1}$. We can prove this by strong induction on n.

Step 1: Base case $n = 1$. We have $a_1 = 1$ since obviously there's only one way to compute x_1, namely do nothing. Since also $C_0 = 1$, the base case holds.

Step 2: Suppose that $n \geq 2$ and that $a_k = C_{k-1}$ for all $1 \leq k < n$. We wish to deduce that $a_n = C_{n-1}$. I claim that the sequence (a_n) satisfies the following recurrence:

$$a_n = \sum_{m=1}^{n-1} a_m a_{n-m}. \tag{0.10}$$

Assuming this is true, and using the strong induction hypothesis, we could deduce that

$$a_n = \sum_{m=1}^{n-1} C_{m-1} C_{n-1-m} = C_{n-1},$$

where the last inequality follows from eq. (8.1) in the lecture notes. Hence it suffices to prove (0.10).

For each $m = 1, \ldots, n-1$, consider all possible ways of computing the product $\prod_{i=1}^{n} x_i$, such that the final multiplication performed is $A \cdot B$, where $A = \prod_{i=1}^{m} x_i$ and $B = \prod_{j=m+1}^{n} x_j$. Note that, because we can't rearrange the x_i, the final multiplication must be of this form, for some $m \in \{1, \ldots, n-1\}$. Since A is a product of m terms with the same rules for multiplication as before, there are a_m possibilities for its computation. Similarly, there are a_{n-m} possible ways to compute B. Hence, by MP, there are $a_m a_{n-m}$ ways to compute the product of all n terms, given the value of m defining the final multiplication. Finally, by AP, we sum over m to find the total number of possible computations, which proves (0.10).

(ii) I claim that $b_n = C_n$ for every $n \geq 1$. It suffices to show that the sequence (b_n) satisfies the same recurrence as (8.1), namely that

$$b_1 = 1, \quad b_n = \sum_{m=1}^{n} b_{m-1} b_{n-m} \;\; \forall\, n \geq 2. \tag{0.11}$$

That $b_1 = 1$ is obvious since, if there are just $2 = 2 \cdot 1$ points, then there's only one way to draw the single chord between them. Now consider a general $n \geq 2$. Label the points $1,\, 2,\, \dots,\, 2n$ in anti-clockwise order, say. Since no two chords can cross, I first claim that the point 1 must be joined to an even-numbered point. For if it were joined to an odd-numbered point, then this chord would partition the remaining $2n - 2$ points into two sets of odd size on either side of the chord. Hence, at least one of the remaining pairs would have to consist of points on opposite sides of this chord, contradicting the condition that no two chords may cross.

So suppose 1 is joined to the point $2m$, where $1 \leq m \leq n$. This chord partitions the remaining $2n - 2$ points into subsets of size $2(m-1)$ and $2(n-m)$ on opposite sides of the chord. The points must be paired off within each subset, with the same condition as before that no two chords can cross. Hence there are, by definition, b_{m-1} and b_{n-m} possibilities respectively for how to pair off the points on the two sides. By MP, there are thus $b_{m-1} b_{n-m}$ possibilities for how to draw all n chords, given that 1 is joined to $2m$. By AP, we sum over $m = 1,\, \dots,\, n$ to get the total number of possible pairings, which proves (0.11).

6. (i) From Theorem 8.3 in the notes we have the recursion

$$S(n,\, k) = S(n-1,\, k-1) + k \cdot S(n-1,\, k). \tag{0.12}$$

Taking $n = 5$ and $k = 3$ we get

$$S(5,\, 3) = S(4,\, 2) + 3 \cdot S(4,\, 3).$$

But from Exercise 12.1.2 in Biggs we also have the formulae

$$S(n,\, 2) = 2^{n-1} - 1, \quad S(n,\, 2) = \binom{n}{2}. \tag{0.13}$$

Hence, $S(4,\, 2) = 2^{4-1} - 1 = 7$ and $S(4,\, 3) = \binom{4}{2} = 6$, so $S(5,\, 3) = 7 + 3 \cdot 6 = 25$.

(ii) In order to list in an efficient manner all the ways to divide up a 5-element set into three parts, we think about how the formulae (0.12) and (0.13) are derived. To get (0.12), we isolate one element, say 1, and consider two cases:

CASE 1: It is in a bin by itself. Then the remaining four elements are divided into two bins. Eq. (0.13) says that the 7 ways of doing this are in $1-1$ correspondence with pairs $\{A,\ A^c\}$ of non-empty subsets of $\{2,\ 3,\ 4,\ 5\}$. Thus, we get the following seven distributions of balls in bins:

$$\begin{array}{ccc} \{2\}, & \{3,\ 4,\ 5\}, & \{1\}; \\ \{3\}, & \{2,\ 4,\ 5\}, & \{1\}; \\ \{4\}, & \{2,\ 3,\ 5\}, & \{1\}; \\ \{5\}, & \{2,\ 3,\ 4\}, & \{1\}; \\ \{2,\ 3\}, & \{4,\ 5\}, & \{1\}; \\ \{2,\ 4\}, & \{3,\ 5\}, & \{1\}; \\ \{2,\ 5\}, & \{3,\ 4\}, & \{1\}. \end{array}$$

CASE 2: Ball 1 is not on its own. We first distribute the remaining four balls into three bins. The $\binom{4}{2}$ ways of doing so are in 1-1 correspondence with the two-element subsets of $\{2,\ 3,\ 4,\ 5\}$, since we must choose which two balls to place in the same bin - the other two bins will then receive one ball each. For each of these six choices, there are three choices for where to place ball 1. This gives the following list of $6 \times 3 = 18$ possibilities:

$$\begin{array}{lll} \{2,\ 3,\ 1\},\ \{4\},\ \{5\}; & \{2,\ 3\},\ \{4,\ 1\},\ \{5\}; & \{2,\ 3\},\ \{4\},\ \{5,\ 1\}; \\ \{2,\ 4,\ 1\},\ \{3\},\ \{5\}; & \{2,\ 4\},\ \{3,\ 1\},\ \{5\}; & \{2,\ 4\},\ \{3\},\ \{5,\ 1\}; \\ \{2,\ 5,\ 1\},\ \{3\},\ \{4\}; & \{2,\ 5\},\ \{3,\ 1\},\ \{4\}; & \{2,\ 5\},\ \{3\},\ \{4,\ 1\}; \\ \{3,\ 4,\ 1\},\ \{2\},\ \{5\}; & \{3,\ 4\},\ \{2,\ 1\},\ \{5\}; & \{3,\ 4\},\ \{2\},\ \{5,\ 1\}; \\ \{3,\ 5,\ 1\},\ \{2\},\ \{4\}; & \{3,\ 5\},\ \{2,\ 1\},\ \{4\}; & \{3,\ 5\},\ \{2\},\ \{4,\ 1\}; \\ \{4,\ 5,\ 1\},\ \{2\},\ \{3\}; & \{4,\ 5\},\ \{2,\ 1\},\ \{3\}; & \{4,\ 5\},\ \{2\},\ \{3,\ 1\}. \end{array}$$

7. (i) Every non-trivial factorisation corresponds to a partition of the set of k distinguishable primes into two non-empty, indistinguishable (since multiplication is commutative) subsets. Hence the number of non-trivial factorisations is $S(k,\ 2) = 2^{k-1} - 1$.

(ii) I claim that the number of non-trivial factorisations is

$$\frac{1}{2}\left[\prod_{i=1}^{k}(e_i + 1) - 2 - \delta_n\right] + \delta_n \tag{0.14}$$

where

$$\delta_n = \begin{cases} 1, & \text{if } n \text{ is a perfect square,} \\ 0, & \text{otherwise.} \end{cases}$$

To see this, imagine that we have two indistinguishable boxes representing the two factors in a non-trivial factorisation. The factorisation is determined by deciding, for each $i = 1,\ \ldots,\ k$, how many powers of p_i to place in each box. There are $e_i + 1$ ways to divide up $p_i^{e_i}$ among the two factors. Hence, there are $\prod_{i=1}^{k}(e_i + 1)$ ways to divide up all the prime factors of n. We disallow the two partitions where all the factors end up in one box. If n is not a perfect square, then $n_1 \neq n_2$ in every non-trivial factorisation, hence we must divide by 2 to take account of the indistinguishability of the boxes. When n is a perfect square, we do likewise except that the unique factorisation where $n_1 = n_2 = \sqrt{n}$ is only counted once. This explains (0.14).