Exercise Session 3 (7/5): Solutions

1. (i) There are an odd number of vertices of odd degree.
(ii) Suppose there were such a graph. There are 8 vertices and three of them have maximum degree 7. Each of these vertices is joined to everything but itself, hence every vertex in the graph must have degree at least 3. But this contradicts the assumption that there exist vertices of degree 1 and 2.
(iii) See Figure O.F.1(iii).
(iv) See Figure O.F.1(iv).

2. (i) Not isomorphic. The graph on the left has girth 3 while the one on the right has girth 4.
(ii) Isomorphic. See Figure O.F.2(ii).
(iii) Not isomorphic. The graph on the left has girth 5 while the one on the right has girth 4.

3. For example,

$$A \to B \to D \to P \to R \to O \to P \to N \to B \to K \to L \to$$
$$\to N \to M \to L \to R \to Q \to D \to C \to B \to E \to D \to$$
$$\to F \to E \to J \to I \to E \to C \to H \to G \to C \to A.$$

(ii) The graph is illustrated in Figure O.F.3(ii). We seek an Euler path starting at vertex 2 and ending at vertex 6. An example of such a path is

$$2 \to 4 \to 8 \to 5 \to 6 \to 7 \to 0 \to 5 \to 4 \to$$
$$\to 3 \to 8 \to 1 \to 0 \to 3 \to 2 \to 1 \to 6.$$

4. A Platonic solid can be identified with a corresponding planar graph, drawn on a sphere, in which each face of the solid becomes an enclosed region. On the sphere, Euler's formula becomes

$$V - E + R = 2. \tag{0.1}$$

Firstly, by definition,

$$R = n. \tag{0.2}$$

Now the idea is that, for a Platonic solid, V and E can both be expressed in terms of R. First consider edges. Consider all pairs $(\varepsilon,\, \rho)$, where ε is an edge bounding the region ρ. On the one hand, since each region is an e-gon, there must be ne such pairs. On the other hand, each edge marks the boundary between two regions, so the number of pairs must be $2E$. Thus

$$E = \frac{ne}{2}. \tag{0.3}$$

Next consider vertices. Consider this time all pairs $(v,\, \rho)$, where v is a vertex on the boundary of region ρ. On the one hand, since each region is an e-gon, there are ne such pairs. On the other hand, each vertex has degree d, hence belongs to d pairs, and so the number of pairs must be Vd. It follows that

$$V = \frac{ne}{d}. \tag{0.4}$$

Substituting (0.2), (0.3) and (0.4) into (0.1) we get

$$\frac{ne}{d} + \frac{ne}{2} + n = 2 \Rightarrow e\left(\frac{1}{2} - \frac{1}{d}\right) = 1 - \frac{2}{n}. \tag{0.5}$$

The crucial point is that

$$0 < e\left(\frac{1}{2} - \frac{1}{d}\right) < 1. \tag{0.6}$$

From the left, this just implies that $d \geq 3$, which we knew already. From the right, it follows that $e < \frac{2d}{d-2}$. If $d \geq 6$, this implies that $e \leq 2$, which is impossible - a closed polygon must contain at least three edges. Hence $3 \leq d \leq 5$. These cases are now analysed separately:

Case 1: $d = 5$. From (0.6) we have $e < \frac{10}{3}$, hence $e = 3$. Plugging into (0.5) gives $n = 20$. From (0.2), (0.3) and (0.4) we conclude that $V = 12$, $E = 30$, $R = 20$. This is the icosahedron.

Case 2: $d = 4$. From (0.6) we have $e < 4$, hence $e = 3$ again. Plugging into (0.5) gives $n = 8$. From (0.2), (0.3) and (0.4) we conclude that $V = 6$, $E = 12$, $R = 8$. This is the octahedron.

Case 3: $d = 3$. From (0.6) we have $e < 6$, so now there are three possibilities: $e \in \{3, 4, 5\}$. We analyse these three subcases in turn.

Subcase 3.1: Take $e = 3$. Plugging into (0.5) gives $n = 4$. From (0.2), (0.3) and (0.4) we conclude that $V = 4$, $E = 6$, $R = 4$. This is the tetrahedron.

Subcase 3.2: Take $e = 4$. Plugging into (0.5) gives $n = 6$. From (0.2), (0.3) and (0.4) we conclude that $V = 8$, $E = 12$, $R = 6$. This is the cube.

Subcase 3.3: Take $e = 5$. Plugging into (0.5) gives $n = 12$. From (0.2), (0.3) and (0.4) we conclude that $V = 20$, $E = 30$, $R = 12$. This is the dodecahedron.

5. (a) There exist Hamiltonian cycles, e.g.:

$$a \to g \to k \to i \to h \to b \to c \to d \to j \to f \to e \to a.$$

(b) There exist Hamiltonian cycles, e.g.:

$$a \to d \to b \to e \to g \to j \to i \to f \to h \to c \to a.$$

(c) There exist Hamiltonian cycles, e.g.:

$$a \to b \to e \to f \to g \to c \to d \to i \to h \to a.$$

(d) There exist Hamiltonian paths, e.g.:

$$a \to c \to e \to b \to d \to f \to g.$$

However, it can be checked that there are no Hamiltonian cycles. Basically, the reason is that too few paths avoid e. Suppose we start from a (in a Hamiltonian cycle it doesn't matter where one starts). Then one can check that it is not possible to visit every vertex and get back to a without visiting e at least twice.

(e), (f) There exist Hamiltonian paths in the 3×5 grid, e.g.:

$$a \to b \to c \to d \to e \to j \to i \to$$
$$\to h \to g \to f \to k \to l \to m \to n \to o.$$

There exist Hamiltonian cycles in the 4×5 grid, e.g.:

$$a \to b \to c \to d \to e \to j \to i \to h \to g \to l \to m \to$$
$$\to\to n \to o \to t \to s \to r \to q \to p \to k \to f \to a.$$

The general theorem here is the following:

Theorem. *In an $m \times n$ rectangular grid there is always a Hamiltonian path, but there is a Hamiltonian cycle if and only if mn is even and* $\min\{m, n\} \geq 2$.

PROOF: It's easy to see that there is always a Hamiltonian path starting at the top left-hand corner and ending at the bottom right-hand corner, namely just zig-zag your way back and forth along the rows. Similarly, if the number of rows is even and there are at least two of them, then, starting from the top left-hand corner, we can zig zag along the rows as before, but this time avoiding the lefthand column in rows 2 to $m-1$. We will arrive at the bottom right-hand corner, and can then go leftwards along the bottom row and back up along the leftmost column to obtain a Hamiltonian cycle. If instead the number of columns is even and greater than one, we can perform the same procedure except we interchange the role of rows and columns (imagine rotating the grid 90 degrees). This proves that a Hamiltonian cycle exists provided mn is even and $\min\{m, n\} \geq 2$. It is also obvious that no Hamiltonian cycle exists if there is just one row or just one column. So it remains to prove there is no Hamiltonian cycle if mn is odd.

Let $G = G_{m,n} = (V, E)$ denote the $m \times n$ grid graph. The crucial point is that this graph is bipartite. Namely, there is a bipartition $V = (V_1, V_2)$ such that V_1 (resp. V_2) contains all vertices in positions (i, j) such that $i + j$ is even (resp. $i + j$ is odd). There are $\lceil \frac{mn}{2} \rceil$ vertices in V_1 and $\lfloor \frac{mn}{2} \rfloor$ vertices in V_2. Hence, V_1 and V_2 contain exactly the same number of vertices if and only if mn is even. But in any bipartite graph $G = (V_1, V_2, E)$, a necessary condition for existence of a Hamiltonian cycle is that $|V_1| = |V_2|$, for any path in the graph must go back and forth between V_1 and V_2. □

6. (i) Let $v_1 \to v_2 \to \cdots \to v_k$ be a simple, directed path of maximum length in a given n-player tournament. We need to show that $k = n$. Suppose not, in other words, suppose there is some vertex v not present in this chain. We consider three cases which, one may check, exhaust all possibilities:

(a) v beat v_1. In this case we can prefix the directed edge $v \to v_1$ to get a longer chain, contradiction.

(b) v_k beat v. In this case we can suffix the directed edge $v_k \to v$ to get a longer chain, contradiction.

(c) there is some $1 \leq i \leq k-1$ such that v_i beat v but v beat v_{i+1}. In this case we can replace the directed edge $v_i \to v_{i+1}$ by the two edges $v_i \to v \to v_{i+1}$ to get a longer chain, contradiction.

(ii) The simplest example would be a 3-player tournament in which player 1 beat player 2, who beat player 3, who beat player 1.

7. (i) First consider the left-hand graph. We can see that $\chi(G) \geq 4$ since the induced subgraph on $\{b, f, h, i\}$ is a K_4, so already this part of the graph requires 4 colors. On the other hand, since G is a plane graph, the Four-Colour Theorem implies that $\chi(G) \leq 4$. Hence $\chi(G) = 4$. If we colour greedily using the given alphabetical ordering of the nodes, then we will get an explicit 4-colouring as follows:

Step	Node	Colour
1	a	1
2	b	2
3	c	1
4	d	2
5	e	3
6	f	1
7	g	2
8	h	3
9	i	4
10	j	3

Next consider the right-hand graph. We can see that $\chi(G) \geq 4$ since D is at the center of a 5-spoked wheel formed by A, B, E, G, F. On the other hand, this graph is also plane. Hence $\chi(G) = 4$. If we colour greedily using the given alphabetical ordering of the nodes, then we will get an explicit 4-colouring as follows:

Step	Node	Colour
1	A	1
2	B	2
3	C	1
4	D	3
5	E	4
6	F	2
7	G	1
8	H	2

(ii) For example,

Step	Node	Colour
1	F	1
2	C	1
3	G	2
4	E	3
5	H	4
6	A	2
7	B	4
8	D	5