Exercise Session 4 (21/5): Solutions

1. (i) *Prim's algorithm:* If we start at the vertex s, say, then Prim's algorithm may choose the following edges in turn. At several steps it can choose randomly between multiple options.

Step	Edge chosen	Weight
1	$\{s,\, v_2\}$	1
2	$\{s,\, v_3\}$	2
3	$\{v_2,\, v_1\}$	2
4	$\{v_2,\, v_4\}$	2
5	$\{v_4,\, v_7\}$	2
6	$\{v_7,\, v_8\}$	1
7	$\{v_7,\, t\}$	1
8	$\{v_4,\, v_5\}$	2
9	$\{v_5,\, v_6\}$	2
Total weight		15

Kruskal's algorithm: If we start at the vertex s, say, then Kruskal's algorithm may choose the following edges in turn. At several steps it can choose randomly between multiple options.

Step	Edge chosen	Weight
1	$\{s,\, v_2\}$	1
2	$\{v_7,\, v_8\}$	1
3	$\{v_7,\, t\}$	1
4	$\{s,\, v_3\}$	2
5	$\{v_2,\, v_1\}$	2
6	$\{v_2,\, v_4\}$	2
7	$\{v_4,\, v_5\}$	2
8	$\{v_5,\, v_6\}$	2
9	$\{v_4,\, v_7\}$	2
Total weight		15

(ii) The algorithm may add edges and labels in the sequence on the next page. At several steps it can choose randomly between multiple options.

Step	Edge added	Label added
1	$\{s, v_2\}$	$l(v_2) := 1$
2	$\{s, v_3\}$	$l(v_3) := 2$
3	$\{v_2, v_1\}$	$l(v_1) := 3$
4	$\{v_2, v_4\}$	$l(v_4) := 3$
5	$\{v_2, v_5\}$	$l(v_5) := 4$
6	$\{v_2, v_6\}$	$l(v_6) := 5$
7	$\{v_4, v_7\}$	$l(v_7) := 5$
8	$\{v_7, v_8\}$	$l(v_8) := 6$
9	$\{v_7, t\}$	$l(t) := 6$

The unique $s \leftrightarrow t$ path in this tree is found by reading backwards from t:

$$s \leftrightarrow v_2 \leftrightarrow v_4 \leftrightarrow v_7 \leftrightarrow t.$$

2. Let $M_0 = \{bg, di\}$ be the given matching. At the first iteration, if we start from the unmatched node a, then we can find the augmenting path $a \to h$. Thus set $M_1 := \{ah, bg, di\}$.

At the second iteration, if we start from the unmatched node c, then we can find the augmenting path $c \to f$. Thus set $M_2 := \{ah, bg, cf, di\}$.

At the third iteration, if we start from the unmatched node e, then we can find the augmenting path $e \to g \to b \to j$. Thus set $M_3 := (M_2 \cup \{eg, bj\}) \backslash \{bg\} = \{ah, bj, cf, di, eg\}$. This is a perfect matching.

3. Let $A = \{b, d, e, f\}$. Then $N(A) = \{g, j, l\}$, so $|N(A)| = 3 < 4 = |A|$ and Hall's condition is not satisfied. Thus the graph has no perfect matching.

4. Let $G = (X, Y, E)$, where X is the set of committees, Y the set of people and an edge represents a member of the corresponding committee. Let $A = \{$växt, natur, djur, pört, stränd$\}$. Then $N(A) = \{$Axel, Berta, Cecilia, Daniela$\}$. So $|N(A)| = |A| - 1$ and $\partial_A = 1$. So a matching can have size at most $|X| - 1 = 6$. An example of a choice of distinct representatives for 6 of the committees is

Växternas vänner: Axel
Naturens nördar: Berta
Djurens djupingar: Daniela
Sjöarnas skönandar: Erik
Korallens kompisar: Frippe
Pörtenas partner: Cecilia.

5. Let $G = (X, Y, E)$ where X is the set of five guests, including Stefan, Y is the set of five possible portions and an edge represents a portion that the corresponding person finds acceptable. The graph is illustrated in Figure O.4.5(S). There are two possible choices of a perfect matching. First note that U can only be matched to $P2$ and Ji only to $P4$. Hence, once we match these, A can only be matched to $P3$. Then S and Jo can be matched to $P1$ and $P5$ in either order.

6. (a) If n is odd, there are none. If n is even there are two, since in a perfect matching we must, as we go round the cycle, alternate between chosen and not-chosen edges. We can swap one lot for the other, thus giving two choices.
(b) If n is odd, there are none. If n is even, there is one since, as we go along the path, we must alternate between chosen and not-chosen edges as before, but this time the first edge must be chosen as otherwise the first vertex would be left unmatched.
(c) Consider a grid of length $n + 2$. Let $x,\ y,\ z,\ w,\ r,\ s$ be the vertices indicated in Figure 0.4.6(S). In a perfect matching we have the following two options:

OPTION 1: Match x and y. Then neither of the edges $\{x,\ z\}$ and $\{y,\ w\}$ is included in the matching. If we remove these two edges along with the chosen edge $\{x,\ y\}$, then we're left with a grid of length $n + 1$ in which it remains to choose a perfect matching. Hence there are a_{n+1} possible perfect matchings in this case.

OPTION 2: x and y are not matched. In a perfect matching we must match x with something, so we have no choice left but to match it with z. Similarly, we must match y with w. But then neither of the edges $\{z,\ r\}$ and $\{w,\ s\}$ is included in the matching. If we remove these two edges along with everything to their right, then we're left with a grid of length n in which it remains to choose a perfect matching. Hence there are a_n possible perfect matchings in this case.

Finally, the addition principle implies that $a_{n+2} = a_{n+1} + a_n$, v.s.v. This is the same recurrence as for the Fibonacci numbers f_n. We can observe directly that $a_1 = 1$ and $a_2 = 2$ - in the latter case we can either choose the top and bottom edges or the left and right edges. Thus $a_1 = f_2$ and $a_2 = f_3$. It follows that $a_n = f_{n+1}$ for every $n \geq 1$ and hence, by eq. (4.7) in the lecture notes, that

$$a_n = \frac{1}{\sqrt{5}} \left(\gamma^{n+1} + (-1)^n \gamma^{-(n+1)} \right), \quad \gamma = \frac{1+\sqrt{5}}{2}.$$

7. We can find the following sequence of augmenting paths:

Step	Augmenting path	Increase in flow
1	$s \to a \to b \to t$	2
2	$s \to c \to d \to t$	2
3	$s \to a \to d \to t$	4
4	$s \to c \to a \to b \to d \to t$	1

The flow at this point is illustrated in Figure O.4.7(S). It has value $f(s,\ a)+f(s,\ c) = 6 + 3 = 9$. The set of vertices reachable from s by an f-augmenting path is $\mathcal{S} = \{s,\ a,\ b,\ c\}$, while the set of unreachable ones is $\mathcal{T} = \{d,\ t\}$. We have

$$c(\mathcal{S},\ \mathcal{T}) = c(a,\ d) + c(b,\ d) + c(c,\ d) + c(b,\ t) = 4 + 1 + 2 + 2 = 9.$$

Hence, we have found a maximum strength flow and a minimum capacity cut.