Tentamen
MMG610 Diskret Matematik, GU

2018-05-30 kl. 14.00–18.00

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Hjälpmedel: Inga hjälpmedel, ej heller räknedosa

To pass requires 60 points, including any points accrued from the two homeworks during VT-18. Preliminarily, 90 points are required for VG. These thresholds may be lowered but not raised afterwards.

Solutions will be posted to the course homepage directly after the exam. The exam will be graded anonymously. Results will be reported in LADOK no later than June 21. A time and place for reviewing the grading will be communicated via mail. Thereafter, the exams will be stored in the Departmental Reception and the student should contact the examiner about an eventual review.

OBS!

Motivate all your solutions !
In Exercise 1, you do *not* need to compute the answers as base-10 numbers.

Exercises

1. For their ill-fated adventure in Kiev on Saturday night. Liverpool named a squad of 18 players, of whom 7 were British, 6 were from other parts of Europe, 3 from remaining parts of Planet Earth and 2 from the DPRK (= Deep Pit of Reckless Keepers). (13p)

(a) In how many ways could a totally clueless coach, who just picks the team completely at random, choose the starting 11 ?

(b) In that case, what is the probability that he picks no Keeper ?

(c) Suppose the coach at least realises that he must pick exactly one Keeper. Suppose also there is a rule which says at least 4 British players must start the game. How many possibilities does this leave for the starting 11 ?

(d) Of the 18 in the squad, 2 are Keepers, 7 are Defenders, 5 are Midfielders and 4 are Attackers. Liverpool usually play 4-3-3, i.e.: Keeper + 4 Defenders + 3 Midfielders + 3 Attackers. How many possibilities would that leave for the starting 11 ?

(e) Just suppose for the sake of argument that there were 40 in the squad, 10 from each of the four "regions". How many possibilities would there be for the starting 11 if all we cared about was how many were chosen from each "region", not the individual identities of the players ?

NOTE: In parts (a)-(d) we *do* care about the individual identities of the starting 11 players.

2. Using the method of generating functions, solve the recursion (12p)

$$a_0 = 1, \;\; a_1 = 2, \quad a_{n+2} = a_{n+1} + 6a_n + 5 \cdot 3^n \;\; \forall\, n \geq 0.$$

OBS! *Zero* points will be awarded for a solution not employing generating functions, even if fully correct otherwise.

Var god vänd!

3. (a) Define what is meant by a *Dyck path of length* $2n$. (1p)

(b) Draw all Dyck paths of length 6. (2p)

(c) Without using generating functions, prove that the number C_n of Dyck paths of length $2n$ is $C_n = \frac{1}{n+1}\binom{2n}{n}$. (9p)

OBS! *Zero* points for a proof which *does* use generating functions, even if otherwise fully correct.

4. (a) Formulate the addition and multiplication principles in probabilistic language. (2p)

(b) Let k, n be positive integers. Prove that if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ then $R(k, k) > n$. (10p)

5. You are referred to the network G_0 in the attached Figure 1. Let G_1 be the underlying graph (i.e.: remove both the arrows and the weights).

(a) Determine a Hamilton cycle, an Euler path and a maximum size matching in G_1. (5p)

(b) Interpreting the weights as distances, implement Dijkstra's algorithm to determine a shortest path in G_0 from s to t. (5p)

OBS! Write down which edge is added and which label is assigned at each step, in table form.

(c) Interpreting the weights as capacities, implement the Ford-Fulkerson algorithm to determine a maximum flow from s to t in G_0 and a corresponding minimum cut. (7p)

OBS! Start from the everywhere-zero-flow and write down which augmenting path you choose and the increase in flow strength at each step, in table form. Then draw the final flow *in full*.

6. Prove that if $G = (V, E)$ is a graph such that $|V| > 2$ and $\deg(v) \geq |V|/2 \ \forall\, v \in V$, then G possesses a Hamilton cycle. (11p)

7. (a) Define what is meant by the *edge chromatic number* $\Phi(G)$ of a graph $G = (V, E)$. (1p)

(b) Prove that if G is bipartite, then $\Phi(G) = \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of a vertex in G. (10p)

8. Implement the Gale-Shapley algorithm on the dataset in the attached Figure 2. Here $X = \{\alpha, \beta, \gamma, \delta\}$ are the boys, $Y = \{A, B, C, D\}$ are the girls and the boys do the proposing. Present your results as a table indicating how the algorithm proceeds: (12p)

Round	Proposals	Rejections	Strings

In the second column write all pairs (x, y) such that x proposes to y in the current round.
In the third column write all pairs (y, x) such that y rejects x in the current round.
In the fourth column write all pairs (y, x) such that y has x on her string after the current round.
Finally, indicate the stable matching M produced by the algorithm.

Lycka till!

Solutions: Diskret Matematik GU, 180530

1. (a) We must choose 11 players from 18, so there are $\binom{18}{11}$ options.

(b) If we choose no Keeper, then we must choose 11 from 16, leaving $\binom{16}{11}$ options. The probability of choosing no Keeper is thus $\binom{16}{11}/\binom{18}{11} = 7/51$.

(c) If we choose exactly one of the two Keepers, then it remains to choose 10 from 16. Of these at least 4 must be Brits, so we can choose either 4,5,6 or all 7 of the 7 Brits and then, respectively, 6,5,4 or 3 of the remaining 9 playerss.

By MP+AP, the total number of possibilities is

$$2 \times \left[\binom{7}{4}\binom{9}{6} + \binom{7}{5}\binom{9}{5} + \binom{7}{6}\binom{9}{4} + \binom{7}{7}\binom{9}{3} \right] = \cdots = 13104.$$

(d) We choose 1 out of 2 Keeepers, 4 out of 7 Defenders, 3 out of 5 Midfielders and 3 out of 4 Attackers. By MP, the number of options is

$$2 \times \binom{7}{4} \times \binom{5}{3} \times \binom{4}{3} = \cdots = 2800.$$

(e) Let x_1, x_2, x_3, x_4 be the number of players chosen from Britain, rest of Europe, rest of Planet Earth and the DPRK respectively. Then $x_1 + x_2 + x_3 + x_4 = 11$ and $0 \leq x_i \leq 10$. If the x_i could be arbitrary non-negative integers, then the number of solutions to the equation would be $\binom{11+4-1}{4-1} = \binom{14}{3} = 364$. We must exclude, however, the 4 solutions where one of the x_i equals 11 and all others are zero. Thus, we are left with $364 - 4 = 360$ options.

2. Call the given recursion (*). Let $A(x) := \sum_{n=0}^{\infty} a_n x^n$. Then,

$$A(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \overset{(*)}{=} 1 + 2x + \sum_{n=2}^{\infty} a_n x^n,$$

$$xA(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n \overset{(*)}{=} x + \sum_{n=2}^{\infty} a_{n-1} x^n,$$

$$x^2 A(x) = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n.$$

Hence,

$$\begin{aligned}(1 - x - 6x^2)A(x) &= [(1+2x) - x] + \sum_{n=2}^{\infty} (a_n - a_{n-1} - 6a_{n-2}) x^n \\ &\overset{(*)}{=} 1 + x + \sum_{n=2}^{\infty} 5 \cdot 3^{n-2} \cdot x^n = 1 + x + \frac{5}{9} \sum_{n=2}^{\infty} (3x)^n \\ &= 1 + x + \frac{5}{9} \frac{(3x)^2}{1-3x} = 1 + x + \frac{5x^2}{1-3x} = \frac{1 - 2x + 2x^2}{1-3x}.\end{aligned}$$

Since $1 - x - 6x^2 = (1-3x)(1+2x)$, we thus have an expression for $A(x)$ as a rational function:

$$A(x) = \frac{1 - 2x + 2x^2}{(1-3x)^2(1+2x)}.$$

The partial fraction decompostion of the right-hand side has the form

$$\frac{1 - 2x + 2x^2}{(1-3x)^2(1+2x)} = \frac{A}{1-3x} + \frac{B}{(1-3x)^2} + \frac{C}{1+2x}.$$

Multiplyting up and out we get

$$1 - 2x + 2x^2 = A(1-3x)(1+2x) + B(1+2x) + C(1-3x)^2 = \cdots = (A+B+C) + x(-A+2B-6C) + x^2(-6A+9C),$$

which leaves us with three linear equations in three unknowns:

$$A+B+C=1, \quad -A+2B-6C=-2, \quad -6A+9C=2.$$

Solving in the normal manner, we get $A=4/15$, $B=1/3$, $C=2/5$. Thus,

$$\begin{aligned} A(x) &= \frac{4/15}{1-3x}+\frac{1/3}{(1-3x)^2}+\frac{2/5}{1+2x} \\ &= \frac{4}{15}\sum_{n=0}^{\infty}3^nx^n+\frac{1}{3}\sum_{n=0}^{\infty}(n+1)3^nx^n+\frac{2}{5}\sum_{n=0}^{\infty}(-2)^nx^n \\ &= \sum_{n=0}^{\infty}\left[3^n\cdot\left(\frac{4}{15}+\frac{n+1}{3}\right)+\frac{2}{5}\cdot(-2)^n\right]x^n, \end{aligned}$$

and so

$$a_n=\left(\frac{3}{5}+\frac{n}{3}\right)\cdot 3^n+\frac{2}{5}\cdot(-2)^n.$$

3. (a) Definitions 7.3 and 7.4 in the lecture notes.
(b) See Figure L.1.
(c) Theorem 7.6, 2nd proof, in the lecture notes.

4. (a) Equations (11.4), (11.5), (11.8) and (11.9) in the lecture notes.
(b) Theorem 11.2 in the lecture notes.

5. (a) An example of a Hamilton cycle is

$$s\to a\to d\to f\to t\to g\to e\to c\to b\to s.$$

The only two nodes of odd degree are s and b, so an Euler path must go between these. An example is

$$\begin{aligned} & s\to a\to f\to d\to a\to b\to s\to c\to g \\ & \to t\to f\to g\to e\to d\to b\to e\to c\to b. \end{aligned}$$

Regarding matchings, G_1 has an odd number of nodes, namely 9 of them, so at most 8 of them can be matched. An example of a maximum size matching is

$$M=\{\{s,\,a\},\,\{b,\,c\},\,\{d,\,e\},\{f,\,g\}\}.$$

(b) Dijkstra's algorithm will proceed as follows. Steps 3 and 4 are interchangeable.

Step	Edge chosen	Label assigned
1	$\{s,\,b\}$	$l(b):=6$
2	$\{s,\,c\}$	$l(c):=7$
3	$\{s,\,a\}$	$l(a):=8$
4	$\{b,\,d\}$	$l(d):=8$
5	$\{b,\,e\}$	$l(e):=9$
6	$\{d,\,f\}$	$l(f):=11$
7	$\{e,\,g\}$ or $\{c,\,g\}$	$l(g):=13$
8	$\{f,\,t\}$	$l(t):=21$

The unique shortest path between s and t thus located is

$$s\to b\to d\to f\to t$$

which has total length 21.

(c) One can find, for example, the following sequence of f-augmenting paths:

Step	Augmenting path	Increase in flow strength
1	$s \to a \to f \to t$	5
2	$s \to b \to e \to g \to t$	3
3	$s \to c \to g \to t$	6
4	$s \to c \to e \to g \to t$	1
5	$s \to b \to d \to f \to t$	2
6	$s \to a \to d \to f \to t$	1

The flow at this stage is illustrated in Figure L.2. Its strength is $f(s, a) + f(s, b) + f(s, c) = 6 + 5 + 7 = 18$. The set of nodes which can be reached from s by an f-augmenting path is $\mathcal{S} = \{s, a, b, c, d, e\}$. Let $\mathcal{T} := V(G_0)\backslash S = \{f, g, t\}$. We have

$$c(\mathcal{S}, \mathcal{T}) = c(a, f) + c(d, f) + c(e, g) + c(c, g) = 5 + 3 + 4 + 6 = 18.$$

So we have found a maximum flow and a minimum cut.

6. Theorem 15.3 in the lecture notes.

7. (a) Definition 19.4 in the lecture notes.

(b) Theorem 19.6 in the lecture notes.

8. The algorithm will proceed as follows:

Round	Proposals	Rejections	Strings
1	$(\alpha, A), (\beta, A), (\gamma, B), (\delta, C)$	(A, β)	$(A, \alpha), (B, \gamma), (C, \delta)$
2	(β, B)	(B, γ)	$(A, \alpha), (B, \beta), (C, \delta)$
3	(γ, C)	(C, δ)	$(A, \alpha), (B, \beta), (C, \gamma)$
4	(δ, A)	(A, α)	$(A, \delta), (B, \beta), (C, \gamma)$
5	(α, B)	(B, β)	$(A, \delta), (B, \alpha), (C, \gamma)$
6	(β, C)	(C, γ)	$(A, \delta), (B, \alpha), (C, \beta)$
7	(γ, A)	(A, δ)	$(A, \gamma), (B, \alpha), (C, \beta)$
8	(δ, B)	(B, α)	$(A, \gamma), (B, \delta), (C, \beta)$
9	(α, C)	(C, β)	$(A, \gamma), (B, \delta), (C, \alpha)$
10	(β, D)		$(A, \gamma), (B, \delta), (C, \alpha), (D, \beta)$

Since no rejections are issued in Round 10, the algorithm now terminates and the girls accept the boys on their strings, thus giving the stable matching

$$M = \{(\alpha, C), (\beta, D), (\gamma, A), (\delta, B)\}.$$