Tentamen
MMG610 Diskret Matematik, GU

2019-01-08 kl. 08.30–12.30

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Hjälpmedel: Inga hjälpmedel, ej heller räknedosa

To pass requires 60 points, including any points accrued from the two homeworks during VT-18. Preliminarily, 90 points are required for VG. These thresholds may be lowered but not raised afterwards.

Solutions will be posted to the course homepage directly after the exam. The exam will be graded anonymously. Results will be reported in LADOK no later than January 29. A time and place for reviewing the grading will be communicated via mail. Thereafter, the exams will be stored in the Departmental Reception and the student should contact the examiner about an eventual review.

OBS!

Motivate all your solutions !
In Exercise 1, for full marks you need to compute the answer as a base-10 number *only* in part (d).

Exercises

1. In MontyPythonLand there are 10 political parties, including the People's Front of Judaea (PFJ) and the Judaean People's Front (JPF). There are 15 seats in parliament, one assigned to each of 15 geographical regions.

(a) In how many ways can the seats be distributed amongst the parties, if it matters which region(s) a party gets its seat(s) from ? (2.5p)

(b) Same question as in (a), except that we only are about how many seats each party gets ? (2.5p)

(c) Under the same conditions as in (a), if the seats are distributed amongst the parties uniformly at random, what is the probability that PFJ and JPF get exactly 5 sets each and that no other party gets more than one seat ? (3.5p)

(d) In the last election, PFJ got 4 seats, JPF got 3 seats and every other party got 1 seat. How many possibilities does this leave for a governing coalition who together have exactly 8 seats, if PFJ and JPF are bitter enemies who cannot be in government together ? (3.5p)

2. (a) State and prove the Erdős-Szekerés theorem. (6p)

(b) It is easy to prove (you don't need to !) that, if p is a prime and x, y are integers satisfying $x^2 \equiv y^2 \pmod{p}$, then $x \equiv \pm y \pmod{p}$. Using this fact (or otherwise), prove that if p is a prime then there exist integers a, b such that $a^2 + b^2 + 1$ is a multiple of p. (5p)

Var god vänd!

3. *Without* using generating functions, solve the recursion (8p)

$$u_0 = u_1 = 1, \quad u_{n+2} = -4u_{n+1} + 5u_n + 3^n + 1 \ \ \forall\, n \geq 0.$$

OBS! *Zero* points will be awarded for a solution employing generating functions, even if fully correct otherwise.

4. (a) Define the Catalan numbers C_n. (2p)

(b) Using generating functions, prove that $C_n = \frac{1}{n+1}\binom{2n}{n}$. (11p)

OBS! *Zero* points will be awarded for a solution *not* employing generating functions, even if fully correct otherwise.

5. You are referred to the network G in Figure 1. Let G^* be the underlying simple graph, when both the arrows and the weights are removed.

(a) For the graph G^* determine (6p)

i. a Hamilton cycle,

ii. a maximum matching,

iii. a minimum set of egdes whose removal yields a graph with an Euler path,

iv. $\chi(G^*)$.

(b) Implement Dijkstra's algorithm to find a shortest path in G from s to t. Indicate the edge chosen and the label assigned at each step, along with the final path. (5p)

(c) Implement the Ford-Fulkerson algorithm to determine a maximum flow from s to t in G and a corresponding minimum cut. (6p)

OBS! Start from the everywhere-zero-flow and write down which augmenting path you choose and the increase in flow strength at each step, in table form. Then draw the final flow *in full*.

6. State and prove Hall's Marriage Theorem. (11p)

7. (a) Define rigorously the concept of a *(bipartite) stable matching*. (3p)

(b) Describe in full the Gale-Shapley algorithm and prove that it always produces a (bipartite) stable matching. (8p)

(c) Furthermore, prove that G-S always produces a stable matching which is optimal for each proposer. (6p)

8. Prove that in any simple graph G one has (11p)

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1},$$

where $\alpha(G)$ denotes the independence number of G.

(HINT/SUGGESTION: Consider a uniformly random ordering of the vertices and a certain event.)

Lycka till!

Solutions: Diskret Matematik GU, 190108

1. (a) 10^{15}.

(b) $\binom{15+10-1}{10-1} = \binom{24}{9}$.

(c) A/B where

$$A = \binom{15}{5} \times \binom{10}{5} \times \frac{8!}{3!}, \quad B = 10^{15}.$$

(d) We have the following three options for a governing coalition:
(i) PFJ plus 4 other parties besides JFP, giving $\binom{8}{4}$ possibilities,
(ii) JFP plus 5 other parties besides PFJ, giving $\binom{8}{5} = \binom{8}{3}$ possibilities,
(iii) All 8 parties besides PFJ and JFP, giving 1 possibility.

Hence the total number of possibilities is

$$\binom{8}{4} + \binom{8}{3} + 1 = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} + 1 = 70 + 56 + 1 = 127.$$

2. (a) Theorem 3.6 in the lecture notes.

(b) If $p = 2$ then $a = 1$, $b = 0$ is a solution. Suppose now p is an odd prime. The given fact implies that, as x runs over all p congruence classes mod p, x^2 will attain $1 + \frac{p-1}{2} = \frac{p+1}{2}$ different values mod p. Hence, each of a^2 and $-1 - b^2$ can attain $\frac{p+1}{2}$ different values mod p. Since there are only p possible values of an integer mod p and $\frac{p+1}{2} + \frac{p+1}{2} > p$, the pigeonhole principle implies that some value mod p must be attained by both a^2 and $-1 - b^2$. In other words, there exist integers a, b such that $a^2 \equiv -1 - b^2 \pmod{p}$, in other words such that $a^2 + b^2 + 1$ is a multiple of p.

3. The characteristic equation is $x^2 = -4x + 5$, which has roots -5 and 1, so the general solution of the corresponding homogeneous equation is

$$u_{h,n} = C_1 \cdot (-5)^n + C_2.$$

Since 1 is already a solution of the homogeneous part, our guess for a particular solution is $u_n = C_3 \cdot 3^n + C_4 n$. Inserting into the recursion,

$$C_3 \cdot 3^{n+2} + C_4(n+2) = [-4C_3\dot{3}^{n+1} + 5C_3 \cdot 3^n + 3^n] + [-4C_4(n+1) + 5C_4 n + 1] \Rightarrow \dots$$
$$\Rightarrow 9C_3 = -12C_3 + 5C_3 + 1 \quad \text{and} \quad 2C_4 = -4C_4 + 1 \quad \Rightarrow$$
$$\Rightarrow C_3 = \frac{1}{16} \quad \text{and} \quad C_4 = \frac{1}{6}.$$

Hence, the solution to the recursion has the form

$$u_n = C_1 \cdot (-5)^n + C_2 + \frac{3^n}{16} + \frac{n}{6}.$$

Insert the initial conditions:

$$n = 0: \quad u_0 = 1 = C_1 + C_2 + \frac{1}{16} \Rightarrow C_1 + C_2 = \frac{15}{16},$$
$$n = 1: \quad u_1 = 1 = -5C_1 + C_2 + \frac{3}{16} + \frac{1}{6} \Rightarrow \dots \Rightarrow -5C_1 + C_2 = \frac{31}{48}.$$

Solving, we eventually get $C_1 = 7/144$, $C_2 = 8/9$. Hence,

$$u_n = \frac{7}{144} \cdot (-5)^n + \frac{8}{9} + \frac{3^n}{16} + \frac{n}{6}.$$

4. (a) C_n is the number of Dyck paths of length $2n$, i.e.: the number of paths in the plane from $(0, 0)$ to $(2n, 0)$, where
- each step of the path is of the form $(x, y) \to (x+1, y \pm 1)$,
- the path never goes below the x-axis.

(b) First proof of Theorem 7.6 in the lecture notes.

5. (a) i. An example of Hamilton cycle is

$$s \to a \to d \to h \to t \to f \to i \to g \to e \to c \to b \to s.$$

ii. Such a matching will have 5 edges. For example,

$$M = \{\{s,\ a\},\ \{b,\ c\},\ \{d,\ e\},\ \{f,\ g\},\ \{h,\ t\}\}.$$

iii. There are 8 vertices of odd degree, 6 of which can be paired off as three edges. Removing these yields a graph with an Euler path and this is the best we can do. So, for example, removing the edges $\{s,\ b\}$, $\{d,\ e\}$ and $\{h,\ t\}$ yields a graph with an Euler path between a and i.

iv. $\chi(G^*) \leq 4$ since G^* is plane. But $\chi(G^*) > 3$ since e is at the centre of an odd cycle formed by $b,\ d,\ f,\ g,\ c$. Hence $\chi(G^*) = 4$.

(b) The algorithm will proceed as follows: Hence the shortest path from s to t is

Step	Chosen arc	Label set
1	$(s,\ b)$	$l(b) := 6$
2	$(s,\ a)$	$l(a) := 7$
3	$(s,\ c)$ or $(b,\ c)$	$l(c) := 8$
4	$(b,\ e)$	$l(e) := 9$
5	$(b,\ d)$	$l(d) := 10$
6	$(e,\ f)$	$l(f) := 11$
7	$(c,\ g)$	$l(g) := 13$
8	$(f,\ h)$	$l(h) := 14$
9	$(f,\ i)$	$l(i) := 16$
10	$(f,\ t)$	$l(t) := 17$

$$s \to b \to e \to f \to t.$$

(c) One can find, for example, the following sequence of f-augmenting paths:

Step	Augmenting path	Increase in flow strength
1	$s \to a \to d \to h \to t$	4
2	$s \to b \to e \to f \to t$	2
3	$s \to c \to g \to i \to t$	5
4	$s \to b \to d \to f \to t$	2
5	$s \to b \to d \to h \to t$	1

The flow at this stage is illustrated in Figure L.1. Its strength is $f(s,\ a) + f(s,\ b) + f(s,\ c) = 4 + 5 + 5 = 14$. The set of nodes which can be reached from s by an f-augmenting path is $\mathcal{S} = \{s,\ a,\ b,\ c,\ d,\ e\}$. Let $\mathcal{T} := V(G)\backslash S = \{f,\ g,\ h,\ i,\ t\}$. We have

$$c(\mathcal{S},\ \mathcal{T}) = c(d,\ h) + c(d,\ f) + c(e,\ f) + c(c,\ g) = 5 + 2 + 2 + 5 = 14.$$

So we have found a maximum flow and a minimum cut.

6. Theorem 18.7 in the lecture notes.

7. (a) See Dataset E.21 and Definition E.22 in the lecture notes.

(b) See Theorem E.23 in the lecture notes.

(c) Theorem E.26 in the lecture notes.

8. Consider a uniformly random ordering of the vertices. For each $v \in V(G)$, let X_v be the indicator of the event E_v that the vertex v appears before all its neighbors. Since the ordering is uniformly random,

$$\mathbb{E}[X_v] = \mathbb{P}(E_v) = \frac{1}{\deg(v) + 1}.$$

Now let $X = \sum_{v \in V(G)} X_v$. In words, X is the number of vertices that appear before all of their neighbors. By linearity of expectaton,

$$\mathbb{E}[X] = \sum_{v \in V(G)} E[X_v] = \sum_{v \in V(G)} \frac{1}{\deg(v) + 1}.$$

In particular, there must be at least *one* way to order the vertices such that the number of them which appear before all their neighbors is at least the above sum. But note that the set of vertices which appear before all their neighbors must form an independent set. Hence, G must possess an independent set of at least this size, Q.E.D.