Tentamen
MMG610 Diskret Matematik, GU

2018-08-24 kl. 08.30–12.30

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Hjälpmedel: Inga hjälpmedel, ej heller räknedosa

To pass requires 60 points, including any points accrued from the two homeworks during VT-18. Preliminarily, 90 points are required for VG. These thresholds may be lowered but not raised afterwards.

Solutions will be posted to the course homepage directly after the exam. The exam will be graded anonymously. Results will be reported in LADOK no later than September 14. A time and place for reviewing the grading will be communicated via mail. Thereafter, the exams will be stored in the Departmental Reception and the student should contact the examiner about an eventual review.

OBS!

Motivate all your solutions !
In Exercise 1, you do *not* need to compute the answers as base-10 numbers.

Exercises

1. A fair, normal dice (i.e.: six sides, numbered $1-6$) is tossed 10 times. (5x2.6p)

An *ordered outcome* refers to the full list of 10 numbers tossed.

An *unordered outcome* records only the number of times each value was tossed. Example: If the ordered outcome was (1, 3, 2, 4, 4, 3, 2, 5, 6, 3), then the unordered outcome would be (1, 2, 3, 2, 1, 1).

In part (a), "probability" refers to *ordered* outcomes.

(a) What is the probability of tossing exactly two 6's ?

(b) Determine the number of ordered outcomes whose corresponding unordered outcome is (3, 0, 3, 2, 2, 0).

(c) Determine the number of ordered outcomes in which every number is tossed either 0, 1 or 3 times.

(d) Determine the total number of possible unordered outcomes.

(e) Determine the number of ordered outcomes for which the sum of all 10 tosses is exactly 52.

2. (a) *Without* using generating functions, solve the recursion (7p)

$$a_0 = 1, \;\; a_1 = 2, \quad a_n = 5a_{n-1} - 4a_{n-2} + 1 \;\; \forall\, n \geq 2.$$

OBS! *Zero* points will be awarded for a solution employing generating functions, even if fully correct otherwise.

(b) Let a_n denote the number of partitions of the set $\{1, 2, \ldots, n\}$ into disjoint subsets, each of which contains either one or two elements. Prove that (5p)

$$a_n = a_{n-1} + (n-1)a_{n-2} \quad \forall\, n \geq 2.$$

Var god vänd!

3. (a) Let $k \in \mathbb{N}$ and $A \subseteq \mathbb{N}$. Define the *k-fold representation function* of A and what it means for A to be an *asymptotic basis of order k*. (2p)

(b) Prove that if $A \subseteq \mathbb{N}$ is an asymptotic basis of order 2, then the 2-fold representation function of A cannot be ultimately constant. (8p)

4. (a) Describe Prim's algorithm for finding a minimal spanning tree in a connected, weighted graph. (2p)

(b) Prove that the algorithm always works. (8p)

5. (a) You are referred to the bipartite graph G_1 in Figure 1. Starting with the single edge $\{5, E\}$, apply the augmenting path procedure to determine a maximum size matching in G_1 in at most four steps. Write down the augmenting path and the new matching chosen at each step. In the first two steps, augment so that the new matching has no edges in common with the previous one (though it may share edges with older matchings). (5p)

Determine also a subset of $X = \{1, 2, 3, 4, 5\}$ of maximum deficiency.

(b) You are referred to the network G_2 in Figure 2. Let G_2' be the underlying graph, when both the arrows and the weights are removed.

i. Determine a Hamilton cycle in G_2'. (1p)

ii. What is the minimum number of edges that need to be added to G_2' in order to obtain a graph (not a multigraph !) with an Euler path ? Add a suitable set of such edges and write down an Euler path in the resulting graph. (2p)

iii. Write out the vertex-coloring of G_2' obtained using the greedy algorithm and ordering the vertices alphabetically. Call your colors $1, 2, 3, 4, \dots$. (2p)

iv. Determine $\chi(G_2')$ and an explicit optimal vertex-coloring. (2p)

v. Implement the Ford-Fulkerson algorithm to determine a maximum flow from s to t in G_2 and a corresponding minimum cut. (5p)

OBS! Start from the everywhere-zero-flow and write down which augmenting path you choose and the increase in flow strength at each step, in table form. Then draw the final flow *in full*.

6. Determine with proof, for every $n \in \mathbb{N}$, the maximum possible number of edges in a triangle-free n-vertex graph. (10p)

7. (a) Define rigorously the concept of a *(bipartite) stable matching*. (3p)

(b) Describe in full the Gale-Shapley algorithm and prove that it always produces a (bipartite) stable matching. (8p)

(c) Furthermore, prove that G-S always produces a stable matching which is optimal for each proposer. (6p)

8. Let m, n be positive integers with $m \leq n$. A *principal* $(m \times m)$*-submatrix* of an $(n \times n)$-matrix $A = (a_{ij})$ is a submatrix consisting of all entries a_{kl}, where $k, l \in S$ for some m-element subset S of $\{1, 2, \dots, n\}$. (11p)

Now fix an $m \in \mathbb{N}$. Prove that if n is sufficiently large, then for any $(n \times n)$-matrix A each of whose entries is either 0 or 1, there exists some principal $(m \times m)$-submatrix B such that

- all the elements below the main diagonal of B are the same, and

- all the elements above the main diagonal of B are the same (though maybe different from those below the diagonal).

Lycka till!