Tentamen
MMG610 Diskret Matematik, GU

2018-08-24 kl. 08.30–12.30

Examinator: Peter Hegarty, Matematiska vetenskaper, Chalmers/GU

Telefonvakt: Sandra Eriksson Barman, telefon: 5325 (alt. Peter Hegarty 070 570 54 75)

Hjälpmedel: Inga hjälpmedel, ej heller räknedosa

To pass requires 60 points, including any points accrued from the two homeworks during VT-18. Preliminarily, 90 points are required for VG. These thresholds may be lowered but not raised afterwards.

Solutions will be posted to the course homepage directly after the exam. The exam will be graded anonymously. Results will be reported in LADOK no later than September 14. A time and place for reviewing the grading will be communicated via mail. Thereafter, the exams will be stored in the Departmental Reception and the student should contact the examiner about an eventual review.

OBS!

Motivate all your solutions !
In Exercise 1, you do *not* need to compute the answers as base-10 numbers.

Exercises

1. A fair, normal dice (i.e.: six sides, numbered $1-6$) is tossed 10 times. (5x2.6p)

An *ordered outcome* refers to the full list of 10 numbers tossed.

An *unordered outcome* records only the number of times each value was tossed. Example: If the ordered outcome was (1, 3, 2, 4, 4, 3, 2, 5, 6, 3), then the unordered outcome would be (1, 2, 3, 2, 1, 1).

In part (a), "probability" refers to *ordered* outcomes.

(a) What is the probability of tossing exactly two 6's ?

(b) Determine the number of ordered outcomes whose corresponding unordered outcome is (3, 0, 3, 2, 2, 0).

(c) Determine the number of ordered outcomes in which every number is tossed either 0, 1 or 3 times.

(d) Determine the total number of possible unordered outcomes.

(e) Determine the number of ordered outcomes for which the sum of all 10 tosses is exactly 52.

2. (a) *Without* using generating functions, solve the recursion (7p)

$$a_0 = 1, \;\; a_1 = 2, \quad a_n = 5a_{n-1} - 4a_{n-2} + 1 \;\; \forall \, n \geq 2.$$

Obs! *Zero* points will be awarded for a solution employing generating functions, even if fully correct otherwise.

(b) Let a_n denote the number of partitions of the set $\{1, 2, \ldots, n\}$ into disjoint subsets, each of which contains either one or two elements. Prove that (5p)

$$a_n = a_{n-1} + (n-1)a_{n-2} \quad \forall \, n \geq 2.$$

Var god vänd!

3. (a) Let $k \in \mathbb{N}$ and $A \subseteq \mathbb{N}$. Define the *k-fold representation function* of A and what it means for A to be an *asymptotic basis of order k*. (2p)

(b) Prove that if $A \subseteq \mathbb{N}$ is an asymptotic basis of order 2, then the 2-fold representation function of A cannot be ultimately constant. (8p)

4. (a) Describe Prim's algorithm for finding a minimal spanning tree in a connected, weighted graph. (2p)

(b) Prove that the algorithm always works. (8p)

5. (a) You are referred to the bipartite graph G_1 in Figure 1. Starting with the single edge $\{5, E\}$, apply the augmenting path procedure to determine a maximum size matching in G_1 in at most four steps. Write down the augmenting path and the new matching chosen at each step. In the first two steps, augment so that the new matching has no edges in common with the previous one (though it may share edges with older matchings). (5p)

Determine also a subset of $X = \{1, 2, 3, 4, 5\}$ of maximum deficiency.

(b) You are referred to the network G_2 in Figure 2. Let G_2' be the underlying graph, when both the arrows and the weights are removed.

i. Determine a Hamilton cycle in G_2'. (1p)

ii. What is the minimum number of edges that need to be added to G_2' in order to obtain a graph (not a multigraph !) with an Euler path ? Add a suitable set of such edges and write down an Euler path in the resulting graph. (2p)

iii. Write out the vertex-coloring of G_2' obtained using the greedy algorithm and ordering the vertices alphabetically. Call your colors $1, 2, 3, 4, \ldots$. (2p)

iv. Determine $\chi(G_2')$ and an explicit optimal vertex-coloring. (2p)

v. Implement the Ford-Fulkerson algorithm to determine a maximum flow from s to t in G_2 and a corresponding minimum cut. (5p)

OBS! Start from the everywhere-zero-flow and write down which augmenting path you choose and the increase in flow strength at each step, in table form. Then draw the final flow *in full*.

6. Determine with proof, for every $n \in \mathbb{N}$, the maximum possible number of edges in a triangle-free n-vertex graph. (10p)

7. (a) Define rigorously the concept of a *(bipartite) stable matching*. (3p)

(b) Describe in full the Gale-Shapley algorithm and prove that it always produces a (bipartite) stable matching. (8p)

(c) Furthermore, prove that G-S always produces a stable matching which is optimal for each proposer. (6p)

8. Let m, n be positive integers with $m \leq n$. A *principal* $(m \times m)$*-submatrix* of an $(n \times n)$-matrix $A = (a_{ij})$ is a submatrix consisting of all entries a_{kl}, where $k, l \in S$ for some m-element subset S of $\{1, 2, \ldots, n\}$. (11p)

Now fix an $m \in \mathbb{N}$. Prove that if n is sufficiently large, then for any $(n \times n)$-matrix A each of whose entries is either 0 or 1, there exists some principal $(m \times m)$-submatrix B such that

- all the elements below the main diagonal of B are the same, and

- all the elements above the main diagonal of B are the same (though maybe different from those below the diagonal).

Lycka till!

Solutions: Diskret Matematik GU, 180824

1. (a) The total number of possibe ordered outcomes is 6^{10}. The number of ways of getting exactly two sixes is $\binom{10}{2} \times 5^8$. Hence, the probability of the latter event is $\frac{\binom{10}{2}\times 5^8}{6^{10}}$.

(b) $\binom{10}{3} \times \binom{7}{3} \times \binom{4}{2} = \frac{10!}{3!\,3!\,2!\,2!}$.

(c) We have the following two possibilities:

CASE 1: Three numbers appear three times each and one number appears once. The number of ways this can happen is $\frac{10!}{(3!)^3} \times \binom{6}{3} \times 3$.

CASE 2: Two numbers appear three times each and four numbers appear once each. The number of ways this can happen is $\frac{10!}{3!\,3!\,4!} \times \binom{6}{2} \times 4!$.

Finally, by AP, the total number of possibilities is

$$\left[\frac{10!}{(3!)^3} \times \binom{6}{3} \times 3\right] + \left[\frac{10!}{3!\,3!\,4!} \times \binom{6}{2} \times 4!\right].$$

(d) An unordered outcome is a sextuple $(x_1, x_2, x_3, x_4, x_5, x_6)$ such that $0 \leq x_i \leq 6$ and $\sum_{i=1}^{6} x_i = 10$. First, if we ignored the restriction $x_i \leq 6$, then the number of possibilities would be $\binom{10+6-1}{6-1} = \binom{15}{5}$. We must subtract those solutions where some $x_i \geq 7$. Note that there can be at most one such i, and there are six choices for it. The remaining five variables then sum to 3, 2, 1 or 0, which is equivalent to six variables summing to exactly 3 (the sixth variable records 3 minus the sum of the others). Hence, the number of solutions to be subtracted equals $6 \times \binom{3+6-1}{6-1} = 3 \times \binom{8}{3}$.

ANSWER: $\binom{15}{5} - 3 \times \binom{8}{3}$.

(e) For $i = 1, \ldots, 10$, let $x_i = 6 - (i\text{:th toss})$. Then $0 \leq x_i \leq 5$ and $\sum_{i=1}^{10} x_i = 8$. Thus it's the same type of exercise as part (d).

ANSWER: $\binom{17}{8} - 10 \times \binom{11}{2}$.

2. (a) The characteristic equation is $x^2 = 5x - 4$, which has roots 1 and 4, so the general solution of the corresponding homogeneous equation is

$$a_{h,n} = C_1 + C_2 \cdot 4^n.$$

Since 1 is already a solution of the homogeneous part, our guess for a particular solution is $a_n = C_3 \cdot n$. Inserting into the recursion,

$$C_3 n = 5C_3(n-1) - 4C_3(n-2) + 1 \Rightarrow \cdots \Rightarrow C_3 = -\frac{1}{3}.$$

Hence, the solution to the recursion has the form

$$a_n = C_1 + C_2 \cdot 4^n - \frac{n}{3}.$$

Insert the initial conditions:

$$n = 0: \quad a_0 = 1 = C_1 + C_2,$$
$$n = 1: \quad a_1 = 2 = C_1 + 4C_2 - \frac{1}{3} \Rightarrow C_1 + 4C_2 = \frac{7}{3}.$$

Solving, we get $C_1 = 5/9$, $C_2 = 4/9$. Hence,

$$a_n = \frac{1}{9}\left(5 + 4^{n+1} - 3n\right).$$

(b) For a partition $\mathcal{P}$ of $\{1, 2, \ldots, n\}$, where $n \geq 2$, there are a priori two options:

CASE 1: n is on its own in a one-element subset. Then it remains to partition the remaining $n-1$ elements under the same conditions as initially, so we have a_{n-1} options.

CASE 2: n is in a two-element subset. There are $n-1$ options for n:s "partner", and then a_{n-2} ways to partition the remaining $n-2$ elements. By MP, we have $(n-1)a_{n-2}$ options in Case 2.

Finally, apply AP.

3. (a) Definition 9.8 and Notation 9.11 in the lecture notes.

(b) Theorem 9.12 in the lecture notes.

4. (a) See notes for Lecture 17, pages 3-4.

(b) Theorem 18.1 in the lecture notes.

5. (a)

Step	Augmenting path	New matching
1	$B \rightsquigarrow 5 \to E \rightsquigarrow 1$	$\{1, E\},\ \{5, B\}$
2	$2 \rightsquigarrow B \to 5 \rightsquigarrow E \to 1 \rightsquigarrow C$	$\{1, C\},\ \{2, B\},\ \{5, E\}$
3	$4 \rightsquigarrow D$	$\{1, C\},\ \{2, B\},\ \{4, D\},\ \{5, E\}$

The table shows a possible solution. An example of a subset of X with deficiency 1 is $A = \{2, 3, 5\}$, since $N(A) = \{B, E\}$. Since the table shows a matching of size $4 = 5 - 1$, this matching must be maximal, and A has maximum deficiency.

(b) i. An example of a Hamilton cycle is

$$s \to a \to b \to d \to e \to g \to f \to h \to i \to t \to j \to c \to s.$$

ii. There are six nodes of odd degree, namely s, a, c, e, h, t. If we join two pairs that are not already joined, then we'll get a graph with an Euler path and this is the best we can do.
For example, add the edges $\{s, e\}$ and $\{h, t\}$. In this new graph, only a and c have odd degree, so there will be an Euler path between them. An example of such a path is

$$a \to s \to b \to a \to f \to d \to b \to e \to d \to g \to f \to h$$
$$\to i \to t \to h \to g \to t \to j \to g \to e \to j \to c \to e \to s \to e.$$

iii. The coloring will be

$$1 : a, c, d, h, t \quad 2 : b, f, i, j \quad 3 : e, s \quad 4 : g.$$

iv. $\chi(G_2') = 3$. Clearly, at least three colors are needed since the graph has triangles. An example of a 3-coloring is

$$1 : a, d, h, j \quad 2 : e, f, s, t \quad 3 : b, c, g, i.$$

v. One can find, for example, the following sequence of f-augmenting paths:

Step	Augmenting path	Increase in flow strength
1	$s \to a \to f \to h \to i \to t$	2
2	$s \to a \to f \to g \to t$	2
3	$s \to b \to e \to g \to t$	2
4	$s \to b \to d \to g \to t$	2
5	$s \to c \to e \to j \to t$	3
6	$s \to c \to j \to g \to t$	1
7	$s \to c \to j \to g \to h \to i \to t$	1

The flow at this stage is illustrated in Figure L.2. Its strength is $f(s, a)+f(s, b)+f(s, c) = 4+4+5 = 13$. The set of nodes which can be reached from s by an f-augmenting path is $\mathcal{S} = \{s, a\}$. Let $\mathcal{T} := V(G_2)\backslash S = \{b, c, d, e, f, g, h, i, j, t\}$. We have

$$c(\mathcal{S}, \mathcal{T}) = c(a, f) + c(s, b) + c(s, c) = 4 + 4 + 5 = 13.$$

So we have found a maximum flow and a minimum cut.

6. Theorem E.10 in the lecture notes.

7. (a) See Dataset E.21 and Definition E.22 in the lecture notes.

(b) See Theorem E.23 in the lecture notes.

(c) Theorem E.26 in the lecture notes.

8. Identify the numbers 1, 2, 3, 4 with the ordered pairs (0, 0), (0, 1), (1, 0), (1, 1) respectively. Given an $(n \times n)$-matrix $A = (a_{ij})$ of 0s and 1s, we can associate to it a 4-coloring of the edges of K_n, whose vertices have been labelled from 1 through n, according to the rule:

Color the edge $\{i, j\}$, *where* $i < j$, *with the color identified with the ordered pair* (a_{ij}, a_{ji}).

Now fix m. By the generalized Ramsey theorem, if n is sufficiently large then, for any A, the associated edge 4-coloring of K_n must induce a monochromatic K_m-subgraph. Let $S \subset \{1, \dots, n\}$ be the labels on the vertices of this subgraph. Then, if you think about it for a minute, the principal $(m \times m)$-submatrix corresponding to S satisfies our requirements.