Övningsproblem

1.1. 22/3 2012

1 Prove the *Dirchlet* box principle: If n objects are put in m boxes, some box must contain at most $\lfloor n/m \rfloor$ objects while some other box must contain at least $\lceil n/m \rceil$ objects.

Everyone cannot be strictly above the average, everyone cannot be strictly below the average. Hence there will always be one who is at least average and one who is at most average.

Now the average is n/m. Thus there is a box with at most n/m balls in it, and a box with at least n/m balls in it. Now the number of balls is an integer. If $n \geq x$ and n an integer then $n \geq \lceil x \rceil$ by definition of the ceiling. Similarly if $n \leq x$ and n an integer then $n \leq \lfloor x \rfloor$, again by definition of floor.

2 The Old Egyptians represented fractions as sum of fractions of type $\frac{1}{n}$. Examples are $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$ or $\frac{16}{63} = \frac{1}{7} + \frac{1}{9}$. Show that we may represent any fraction systematically by using the inductive scheme

$$\frac{m}{n} = \frac{1}{q} + \{\textbf{representations of } \frac{m}{n} - \frac{1}{q}\}\textbf{with } q = \lceil \frac{n}{m} \rceil$$

The point is to show that this inductive procedure terminates. Now if we write $\frac{m}{n} - \frac{1}{q}$ with $q = \lceil \frac{n}{m} \rceil$ we get $\frac{m\lceil \frac{n}{m} \rceil - n}{nq}$ Let us concentrate on the numerator. We can write it as $m(\frac{n}{m} + x) - n = mx$ where $0 \leq x < 1$ as $0 \leq \lceil x \rceil - x < 1$. Hence the new numerator mx is strictly less than the previous. As the numerators will always be positive, there will be a smallest strictly positive numerator, in which case we will have equality, and the process will stop.

Example $\frac{3}{7}$ At the first stage $\frac{3}{7} = \frac{1}{3} + (\frac{3}{7} - \frac{1}{3})$ the next fraction is given by $\frac{2}{21}$ we proceed getting $\frac{2}{21} = \frac{1}{11} + (\frac{2}{21} - \frac{1}{11})$ where the latter fraction simplifies to $\frac{1}{231}$ hence

$$\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$$

3 Show that

$$\lceil \frac{n}{m} \rceil = \lfloor \frac{n+m-1}{m} \rfloor$$

We have $\frac{n}{m} < \frac{n+(m-1)}{m}$ and we need to show that there is exactly one integer N sandwiched between the two. This will show it because by definition we will have $N = \lceil \frac{n}{m} \rceil$ and $N = \lfloor \frac{n+m-1}{m} \rfloor$. Consider the m consequetive numbers $n, n+1, n+2 \ldots n+(m-1)$ exactly one of those will be divisible by m, and any integer N between them will give rise to $n \leq Nm \leq n+(m-1)$.

You may also think of negative remainders. We can write every n as $qm - r$ where $0 \leq r < m$ and then $n + r$ is divisble by m